

Analysis Written Examination

Thursday, August 25, 1994 - 9:00 A.M. - 1:00 P.M.

Gibson Hall, Room 415

PART 1.

1. Let  $f$  be an entire function. For each  $r > 0$ , define

$$\Omega_r = \{z \in \mathbb{C} : |f(z)| > r\}.$$

Show that  $\Omega_r$  is open for each  $r$  and that each connected component of  $\Omega_r$  is unbounded.

2. Let  $f$  be a holomorphic (= analytic) function on the unit disc  $D$  centered at 0. For each  $z \in D$ , prove that

$$|\text{dist}(z, |z| = 1)|^n |f^{(n)}(z)|$$

is bounded as  $|z| \rightarrow 1$ .

3. Compute  $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$ .

4. Find a conformal map from  $\{z \in \mathbb{C} : 0 < \text{Im } z < a\}$  to the disc of radius 2 centered at the point  $1 + i$ .

5. Let  $f$  be an entire function. Suppose that there exists  $R > 0$  such that  $|f(z)| \leq C|z|^n$  for  $|z| > R$ . Prove that  $f(z)$  is a polynomial. What is the maximum degree possible?

## PART 2.

1. Given  $p > 1$ , determine the values of  $\alpha \in \mathbb{R}$  such that the map

$$f \mapsto \int_1^{\infty} \frac{f(x)}{x^\alpha} dx$$

define an element of  $L^p([1, \infty))^*$ .

2. Let  $f(x) = \cos\left(\frac{1}{x}\right)$ ,  $f(0) = \frac{1}{2}$ .

- Is  $f$  Lebesgue integrable on  $[0, 1]$ ?
- Is  $f$  Riemann integrable?
- Is  $f$  of bounded variation?
- Is  $f$  Lipschitz?

This means that  $|f(x) - f(y)| \leq C|x - y|$  for some  $C > 0$ .

3. Let  $X$  be the space of all continuously differentiable functions  $f$  defined on  $\mathbb{R}$  such that

$$\lim_{x \rightarrow \pm\infty} f(x) = \lim_{x \rightarrow \pm\infty} f'(x) = 0.$$

Define  $\|f\| = \max_{x \in \mathbb{R}} |f'(x)|$ . Is  $X$  a Banach space?

4. Does  $\lim_{k \rightarrow \infty} \int_0^{2\pi\sqrt{2k}} \cos^{2k}\left(\frac{x}{\sqrt{2k}}\right) dx$  exist?

5. Let  $E \subseteq \mathbb{R}$  be a measurable set. For  $x \in E$  and  $h > 0$ , let

$$E(x, h) = E_r \cap [x - h, x + h].$$

Show that for almost all  $x \in E$

$$\lim_{h \rightarrow 0} \frac{m(E(x, h))}{2h} = 1.$$