Analysis Preliminary Exam
Friday January 6 1995

1. Let $\{f_n\}$ be a sequence of integrable functions on the measure space (X, A, μ) which converges a.e. to an integrable function f. Is it always true that

$$\int f \, d\mu \le \liminf \int f_n \, d\mu \ ?$$

Prove it or give a counterexample.

2a. Give an example of a sequence $\{h_n\}$ of functions in L^1 on some measure space (X, A, μ) which converges a.e. to an integrable function h, but which does not converge in the L^1 distance to h.

Let $\{f_n\}$ be a sequence of functions in $L^1(X, A, \mu)$ which is Cauchy in L^1.

b. Is the sequence necessarily Cauchy in measure?

c. Suppose in addition that $f_n \to g$ a.e. Does $f_n \to g$ in L^1?

3. Let $f_n \ge 0$ be a sequence of integrable functions on (X, A, μ), n = 1, 2, ... Prove that

$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu$$

if the right side is finite.

4. Let H be a Hilbert space with inner product $\langle \, , \, \rangle$ in the real numbers R and with complete orthonormal system $\{v_0, v_1, \ldots\}$.

a. If $v, w \in H$ satisfy $\| v + w \| = \sqrt{2}$, $\| v - w \| = 1$, find $\langle v, w \rangle$.

b. If $v, w \in H$ satisfy $\langle v, v_n \rangle = 1/(2n+1)$, $\langle w, v_n \rangle = (-1)^n (1/\sqrt{3})^{2n+1}$, estimate $\langle v, w \rangle$ to within 1/100 of its exact value.

c. Let H be the Hilbert space $L^2([0, 1])$ with respect to Lebesgue measure. Specify a complete orthonormal system $v_0, v_1, v_2, \ldots$ for H, and give a series expansion for an element $v \in H$, with $\langle v, v_n \rangle = 1/(2n+1)$.

5. Give an example of a function on a measure space (X, A, μ) which is in L^1 but which is not in L^2.

6. Let $R = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0, \mathrm{Re}(z) < 0\}$, and let H be the upper half plane $H = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$.

a. Find a conformal map f from R onto H.

b. Let $u(x, y)$ on H be given by $u(x, y) = x$, so grad(u) is perpendicular to the boundary normal vector $\mathbf{n} = (0, 1)$. Find the harmonic conjugate of u (the function v such that $\Phi(x + iy) = u(x, y) + iv(x, y)$ is analytic on H).

c. Find a nonconstant function $u_R(x, y)$ on the closure of R which is harmonic ($\Delta u_R = 0$) on R, continuous on the closure of R, and such that at the boundary of R its nonzero gradient is perpendicular to the boundary normal vector $\mathbf{n}_R$.

7. Let f be analytic on the open disk $D = \{z \in \mathbf{C} : |z| < 1\}$, and suppose that $|f| \le 1$ on D.

a. Find a bound on $|f'(0)|$.

b. Find a function f for which your bound cannot be improved.

c. Suppose in addition that $f(0) = i$. Find $\lim_{z \to i} f^2(z)$, where z approaches i from inside D.

8. Compute $\displaystyle\int_{|z|=1/2} \frac{1}{(1-iz)z}\, dz$.

9. Give a convergent series for the residue $\mathrm{Res}(f, 0)$ at 0 of the function $f(z) = e^{z + 1/z}/z$, and estimate $\displaystyle\frac{1}{2\pi i}\int_{|z|=1} f(z)\, dz$ to within 1/100 of its exact value.