

January 6, 1999

Instructions: There are 12 problems in this exam. Each problem worth 10 points. Only the best 5 solutions from part 1 and best 5 solutions from part 2 will be counted.

Part 1. Complex Analysis

1. Assume that the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < R$.
 - (a). Prove that if $0 < r < R$ then the series converges absolutely and uniformly for $|z| \leq r$.
 - (b). Prove that the series $\sum_{n=0}^{\infty} n a_n z^n$ also converges for $|z| < R$.
2.
 - (a). Give an explicit conformal map from the half plane $\{z \mid \operatorname{Re} z > 0\}$ onto the unit disk $\{z \mid |z| < 1\}$.
 - (b). Suppose that an entire function $f(z)$ satisfies $\operatorname{Re} f(z) \neq 0$ for all z in the complex plane. Prove that $f(z)$ is a constant function.
3. Use contour integration to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{e^{itx}}{x^2 + 1} dx$$

for t a real constant.

4. Let $f(z) = \frac{z}{\sin z}$. We have $f(z) = \sum_{n=0}^{\infty} c_n z^n$ in a neighborhood of $z = 0$.
 - (a) Evaluate c_0 .
 - (b) Evaluate c_1 .
 - (c) Find the radius of convergence of this power series.
5. Let $t > 1$ be a real constant. Show that the equation $z + e^{-z} = t$ has exactly one root in the half plane $\{z \mid \operatorname{Re} z > 0\}$ and that this root is a real number.
6. Suppose that $f(z)$ is an entire function satisfying $|f(z)| \leq (1 + |z|)^n$ for a positive integer n and for all z in the complex plane. Show that f is a polynomial function of degree $\leq n$.

Part 2. Real Analysis

7. For a continuous function f on $(0, 1)$, let $\|f\|_p$ be the L^p -norm of f for $p \geq 1$.
 - (a) Give an example of an unbounded continuous function g on $(0, 1)$ such that the L^p -norm of g is finite for all $1 \leq p < \infty$.

- (b) Suppose that g is such an example which satisfies the properties of part (a). Find $\lim_{p \rightarrow \infty} \|g\|_p$.
8. Let f be a continuous function on $[0, 1]$.
- (a) Find $\lim_{n \rightarrow \infty} n \int_0^1 x^{2n} f(x) dx$.
- (b) Prove your assertion in part (a).
9. Suppose that f and g are positive measurable functions on $[0, 1]$ such that $f(x)g(x) \geq 1$ almost everywhere. Prove that $\int_0^1 f(x) dx \int_0^1 g(x) dx \geq 1$.
10. Suppose that $\{c_n\}_{n=1}^{\infty}$ is a sequence of real numbers and $\sum_{k=1}^{\infty} |c_k| < \infty$. Let $f_n(x) = \sum_{k=1}^n c_k \cos(kx)$. Show that
- (a) f_n converges pointwise to a function f .
- (b) f_n converges to f uniformly.
- (c) the function f is continuous.
11. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a nonnegative valued continuous function. Suppose that g vanishes outside the interval $[-1, 1]$ and satisfies $\int_{-\infty}^{\infty} g(y) dy = 1$. Prove that $\lim_{k \rightarrow \infty} k \int_{-\infty}^{\infty} f(x+y)g(ky) dy = f(x)$ for all x .
12. Suppose that $f \in L^1(\mathbb{R})$ is a Lebesgue integrable function on the real line \mathbb{R} and g is a C^∞ -function with compact support (i.e., g is a C^∞ -function and vanishes outside some compact set). Let

$$h(x) = \int_{-\infty}^{\infty} g(x-y)f(y) dy$$

- (a) Show that h is a bounded function.
- (b) Show that h is differentiable.
- (c) Show that h is a C^∞ -function.
- (d) Use a well known double integral theorem to show that h is in $L^1(\mathbb{R})$, i.e., h is Lebesgue integrable on \mathbb{R} .