Analysis Exam
August 26, 2003

Instructions: There are three sections to the exam:Real Analysis, Functional Analysis and Complex Analysis. You are asked to answer two questions from each part. You may use any result from Mathematics 630,721 and 722 without proof.

1.Real Analysis

1.1 a) Define what it means for a sequence, $\{f_n(x)\}$, of functions defined on a measure space (X, μ) to converge (i) in measure, (ii) almost everywhere, (iii) uniformly, (iv) in $L^1(X, \mu)$.
b) In a measure space of finite measure, which modes of convergence above imply which others?

1.2 Suppose that $f(x, y)$ is integrable on the square $S = [0,1] \times [0,1]$ and define

$$F(x, y) = \int_0^x \int_0^y f(u, v)\, du\, dv.$$

Show that $\frac{\partial^2 F}{\partial x \partial y}$ exists almost everywhere on S and is almost everywhere equal to f.

1.3 Suppose f_n is a sequence of uniformly bounded functions the interval $I = [0,1]$ with the property that each f_n is continuous on $I \setminus Z_n$ where the Lebesgue measure of Z_n is zero. Suppose that $f_n \to g$ uniformly on I for some function g. Show that g is Riemann integrable.

1.4 Consider the set,$\mathcal{S}$, of (equivalence classes of) bounded measurable functions on the interval $[0, 1]$. For $u(t) \in \mathcal{S}$ define

$$\|u\| = \int_0^1 \frac{|u(t)|}{1 + |u(t)|}\, dt.$$

a) Show that this defines a metric on the space of bounded measurable functions. It may be helpful to observe the inequality of real numbers,

$$\frac{A + B}{1 + |A + B|} \leq \frac{|A|}{1 + |A|} + \frac{|B|}{1 + |B|}.$$

b) Show that in this metric space a sequence $u_n(t)$ converges to $u(t) \in \mathcal{S}$ if and only if $u_n(t)$ converges to $u(t)$ in measure.
c) Use what you know about convergence in measure to show that, with this metric, the space $\mathcal{S}$ is complete.

2.Functional Analysis

2.1 Let λ_j be a sequence of complex numbers none of which is equal to 1 and with $\lim_{j\to\infty} \lambda_j = 1$. Let H be a separable hilbert space with orthonormal basis $\{e_k\}_{k=1}^{\infty}$ and define $T : H \to H$ by $Te_i = \lambda_i e_i$. Prove

(a) The point spectrum of T consists exactly of the points λ_i.

(b) The residual spectrum of T is empty.

(c) The continuous spectrum of T consists of the point 1.

(d) The resolvent set of T consists of $\mathbb{C} \setminus \cup_{i=1}^{\infty} \{\lambda_i\} \setminus \{1\}$.

✓2.2 Suppose that $\{x_k\}$ is a sequence of real numbers such that $\sum_n x_n^2 < \infty$ which has the property that for any other sequence $\{y_k\}$ which has the same property (i.e. $\sum_k y_k^2 < \infty$ the sum

$$\sum_{k=1}^{\infty} x_k y_k = 0.$$

Show that $x_k = 0$ for all k.

✓2.3 Let X be a Banach space with dual space X^*. Define a map

$$i : X \to X^{**}$$

by $i(x)(f) = f(x)$ for $f \in X^*$ and $x \in X$. Show that i is one to one.

2.4 Suppose that f_n is a sequence of continuous functions on $[0, 1]$ with the property that

$$\sup_{x\in[0,1]} |f(x)| + \sup_{x\neq y\in[0,1]} \frac{|f(x) - f(y)|}{|x - y|} \leq C.$$

Show that some subsequence of the f_n converges uniformly/

3.Complex Analysis

✓3.1 Compute

$$\int_0^{\infty} \frac{\cos 2x}{x^2 + 1}\, dx$$

using contour integration.

3.2 Suppose that $f(z)$ is an entire holomorphic function which maps the real axis onto itself and the imaginary axis onto itself. Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is the power series representation of f about the origin. Show

(a) $a_n \in \mathbb{R}$ for all n,
(b) $a_{2n} = 0$,
(c) f is an odd function.

3.3 Suppose that $f(z)$ is a holomorphic function which maps the right half plane $\{z \mid Re(z) > 0\}$ into the unit disc with $f(2) = 0$. Show that $|f(1)| \leq \frac{1}{3}$.
Hint: The map $T(z) = \dfrac{2(1+z)}{1-z}$ maps the unit disc to the right half plane taking the origin to $z = 2$. Apply the Schwarz lemma to the function $f(T(z))$.

3.4 Suppose that $f(z)$ is an entire function with the property that $|f(z)| \leq \sqrt{R}$ for $|z| = R$. Show that f is a constant function.