

ANALYSIS QUALIFYING EXAM
August 23, 2004

Directions: Do as many problems as possible, some from each part.

PART A. (functions complex valued, unless otherwise specified)

1. Use the residue calculus to evaluate these definite integrals. Of course they can be done by elementary means, but the point here is to show your grip on the residue calculus.

a) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

b) $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$

2. Find a fractional linear transformation (Moebius transformation) carrying $\{z \in \mathbb{C} : |z| < 1\}$ onto the upper half plane.
3. State and prove Liouville's Theorem.
4. Suppose f is holomorphic in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Suppose $|f(z)| \leq 1$ for all $z \in \Delta$, and $f(0) = f'(0) = 0$.
- (a) Find (with justification) an upper bound for $f(1/2)$.
- (b) Show by example your bound is best possible.
5. Prove or disprove:
- a) There is a nontrivial function holomorphic on all of \mathbb{C} with $f(0) = 0 = f^{(k)}(0)$ for all positive integers k .
- b) There is a nontrivial real valued C^∞ function f on \mathbb{R} with $f(0) = 0 = f^{(k)}(0)$ for all positive integers k .

PART B. (all functions real valued)

6. On \mathbb{R} , define

a) Borel set b) Lebesgue measurable set.

Which is the more general notion? Explain fully, including how to produce sets in the more general class which are not in the more special class.

7. Suppose $\{f_n\}$ is a sequence of Lebesgue measurable functions on $[0,1]$ such that

$f_n \rightarrow 0$ a.e. Show that for each $\varepsilon > 0$ we have $m\{x : |f_n(x)| > \varepsilon\} \rightarrow 0$ as $n \rightarrow \infty$. Here m is Lebesgue measure. (You should give a proof based on a convergence theorem and not simply appeal to facts concerning convergence in measure.)

8. State the Lebesgue version of the Fundamental Theorem of Calculus. Define all relevant notions.

9. Prove (by citing appropriate theorems) or disprove (by example).

a) If $\{f_n\}$ is a sequence of continuous functions on $[0,1]$ such that $f_n \rightarrow 0$

pointwise, then $\int_0^1 f_n(x) dx \rightarrow 0$.

b) If $\{f_n\}$ is a sequence of continuous functions on $[0,1]$ such that $|f_n(x)| \leq 1$ for

all n, x and such that $f_n \rightarrow 0$ pointwise, then $\int_0^1 f_n(x) dx \rightarrow 0$.

10. Suppose $\{A_n\}$ is a sequence of sets such that $[0,1] = \bigcup A_n$.

a) If each A_n is Borel, show that at least one A_n has positive Lebesgue measure.

b) If each A_n is closed, show that at least one A_n has nonempty interior.