(I)

Analysis qualifying exam

August 26, 2005.

Please write only your assigned number on the top right corner of your pages. This will ensure anonimity in the grading.

The first five questions of this exam are in Complex Analysis, the last seven are in Real Analysis. These two parts will be graded independently and you must pass both of them.

(1) Evaluate the definite integral

$$\int_0^\infty \frac{\cos x}{1+x^2}\,dx.$$

(2) Let f be an entire function that satisfies $\operatorname{Re} f(z) > 4$. Prove that f is constant.

(3) Let g be a complex valued function that is analytic in a region $\Omega \subset \mathbb{C}$. Assume that $g = u + iv$ with u, v real valued. Prove the orthogonality of the level curves

$$\begin{aligned} C_u &= \{(x,y) \in \mathbb{R}^2 : u(x,y) = C_1\} \\ C_v &= \{(x,y) \in \mathbb{R}^2 : v(x,y) = C_2\}. \end{aligned}$$

(Here C_1 and C_2 are constants). Give an example.

(4) Assume that f is a complex valued function that is analytic with finitely many exceptions $\{z_1, z_2, \cdots, z_n\}$. At $z = z_j$ the function f has a pole of multiplicy $m_j > 0$. Suppose that the function f satisfies the bound $|f(z)| \leq C(1+|z|)^m$ for $|z|$ sufficiently large, say $|z| > R$. What can you say about f?

(5) Let Ω be the region inside

$$C_1 = \{z \in \mathbb{C} : |z-1| = 1\}$$

and outside

$$C_2 = \{z \in \mathbb{C} : |z - \tfrac{3}{2}| = \tfrac{1}{4}\}$$

Is it possible to map Ω conformally onto a region bounded by two concentric circles centered at 0? If yes, find this map.

(6) a) Let $X = C[0,1]$ with the supremum norm. Define the function $\varphi(f) = f(\frac{1}{2})$. Is this a bounded linear functional on X? What is its norm?

b) Let $Y = L^2[0,1]$ with the usual norm. Consider the same functional φ as in part a). Prove that this is not well-defined. Find a dense set $D \subset Y$ on which φ is defined. Is φ a continuous functional on D?

(7) Let $\epsilon > 0$ be fixed. Compute the norm of the operator

$$T(u) = \frac{u(\epsilon) - u(-\epsilon)}{\epsilon}$$

acting on $C[-1,1]$ with the supremum norm. Describe the kernel of T.

(8) Let $Y = L^p(\mathbb{R})$, with $p > 1$. Define

$$T(f,a) = f_a$$

where $f_a(x) = f(x-a)$.

a) Is T a continuous map in the variable f? Give a precise definition of what this means.

b) Give a definition of what it means for T to be a continuous function of $a \in \mathbb{R}$. Is this true?

(9) The Fourier transform of a function is *formally* defined by the integral formula

$$\mathbb{F}(g) = \int_{-\infty}^{\infty} g(k) e^{2\pi i k x}\, dk$$

for functions $g \in L^1(\mathbb{R})$.

a) Assume g is a differentiable function of compact support. Find an expression for the Fourier transform of g'. Justify all your steps. What can you say about the smoothness of the transformed function?

b) Evaluate the Fourier transform of the gaussian

$$g(k) = e^{-ak^2}, \quad a > 0.$$

(10) Consider the function

$$H(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Define

$$f_0(x) = H(x), \ f_1(x) = H(2x) \ \text{ and } f_2(x) = H(4x).$$

Compute the orthonormal projection of $f(x) = x$ onto the subspace of $L^2[-1,1]$ spanned by f_0, f_1, f_2.

(11) Explain the relation between a Borel set and a Lebesgue measurable set. Give a good definition of Lebesgue measurable set on $\mathbb{R}$ and prove that if A and B are disjoint open sets, then $|A \cup B| = |A| + |B|$.

(12) Evaluate

$$\lim_{n\to\infty} \int_0^1 (x + \tfrac{1}{n})^{-1/2}\, dx$$

Justify all your steps.