Analysis Comprehensive Exam Jan 7, 2005

COMPLEX ANALYSIS:

1. Use the method of residues to evaluate $\int_{-\infty}^{\infty} \frac{\exp(ix)}{2-2x+x^2}\,dx$. Then use this to find $\int_{0}^{\infty} \frac{\cos(x)}{2-2x+x^2}\,dx$.
2. Suppose f is holomorphic in $\mathbb{C}$ and $|f(z)| \le |z|$ for all $z \in \mathbb{C}$. Show that $f(z) = cz$ for some constant c with $|c| \le 1$.
3. Suppose f is analytic on the open unit disk and continuous on the closed unit disk. Suppose that $|f(z)| \le M$ for all z with $|z| = 1$, for some constant M. Find an upper bound for $|f^{(2)}(0)|$, and show by example that your upper bound is best possible.
4. Suppose g is holomorphic in the complex plane and $f(z) = g(z) + \frac{1}{z-1-i}$ for all z. Consider the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for f. What can be said about the set of z for which this series converges?

REAL ANALYSIS:

1. For each of the statements (a) and (b) indicate whether the statement is true or false and as appropriate either sketch a proof that the statement is true or indicate how to get a counterexample to the statement:

   (a) If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and $B \subset \mathbb{R}$ is a Borel set, then $f^{-1}(B)$ is Lebesgue measurable.

   (b) If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable and $B \subset \mathbb{R}$ is Lebesgue measurable, then $f^{-1}(B)$ is Lebesgue measurable.
2. Suppose $\{B_n\}$ is a sequence of Borel subsets of $\mathbb{R}$ and m is Lebesgue measure on $\mathbb{R}$. For each statement (a) and (b) indicate whether the statement is true or false and as appropriate either give a proof that the statement is true or give a counterexample. In case the statement is false, add a simple condition making the statement true.

   (a) If $B_{n+1} \subset B_n$ for all n then $\lim_{n\to\infty} m(B_n) = m\left(\bigcap_{n=1}^{\infty} B_n\right)$.

   (b) If $B_n \subset B_{n+1}$ for all n then $\lim_{n\to\infty} m(B_n) = m\left(\bigcup_{n=1}^{\infty} B_n\right)$.

3. A set $S \subset \mathbb{R}$ is totally disconnected if the following condition holds: whenever $a, b \in S$ with $a < b$, then there is some $c \notin S$ with $a < c < b$. Prove or disprove: $\mathbb{R} = \bigcup_{n=1}^{\infty} S_n$ for some sequence $\{S_n\}$ of disjoint closed totally disconnected sets.

4. Consider the real Hilbert space $L^2[0,1]$ with the usual inner product, and let V be the subspace spanned by the constant function 1 and the identity function $f(x) = x$. Of course V is closed. For arbitrary $g \in L^2[0,1]$ find constants c_0, c_1 so that $c_0 1 + c_1 f$ is the orthogonal projection of g onto V.

5. Suppose X is a normed linear space over the real numbers and $F : X \to \mathbb{R}$.

(a) Define what it means for F to be a bounded linear functional.

(b) Show that a linear functional $F : X \to \mathbb{R}$ is bounded if and only if it is continuous everywhere if and only if it is continuous at 0.

(c) If $X = L^2[0,1]$ and $F : X \to \mathbb{R}$ is a bounded linear functional, what can we say about F?

6. (a) Let $V = C[0,1]$ be the space of continuous functions on $[0,1]$ equipped with the max norm. Show that the subspace $C^\infty[0,1]$ of infinitely differentiable functions are dense in V.

(b) Show that V contains a countable dense subset.

(c) Show that $L^\infty[0,1]$ does not contain a countable dense subset. Suggestion: Find an uncountable family of functions in $L^\infty[0,1]$ such that for any two distinct elements f, g in the family, $\|f - g\|_\infty = 1$.