Analysis Qualifying Exam
October 18, 2006

Note: You must pass both parts—complex analysis and real analysis—to pass the exam.

Complex Analysis

1. Evaluate $\oint_{\gamma} q(z)dz$, where $q(z)=\frac{\exp(z)}{1+z^2}$, and γ is the circle of radius 3, centered at 0, traversed counterclockwise.

2. Suppose f is an entire function such that for some constant C we have $|f(z)| \le 1+C|z|^3$ for all z. Say as much as you can about f.

3. Let $f(z)=\frac{1}{1+z^4}\ \forall z \in \mathbb{C}.$ for all $z \in \mathbb{C}$. Suppose we expand $f(z)$ in a power series $\sum_{n=0}^{\infty} a_n z^n$ centered at $z = 1/2+i$. Describe the region of convergence of this power series. Do not worry about boundary behavior.

4. Suppose f is a function continuous on the closed unit disk $D=\{z \in \mathbb{C} : |z| \le 1\}$ and holomorphic in the interior. Suppose that $f(0)=0$ and $|f(z)| \le 1$ for all z with $|z| \le 1$.

(a) Prove that $|f(z)| \le |z|$ for all $z \in D$. What is the name of this result? You should name the result and prove (a).)

(b) What is the greatest possible value of $|f'(z)|$? For which functions is this maximum attained? Again you may quote a result but be sure also to prove (b)!

Real Analysis

1. Consider the unit interval [0,1]. In this context,
(a) define Lebesgue measurable set
(b) Define Lebesgue measurable function.

2. Suppose f is a continuous function on [0,1]. Show that f is uniformly continuous on [0,1].

3. (a) State the Lebesgue Dominated Convergence Theorem for integrable functions on $[0,1]$.

   (b) Give an example of continuous functions $\{f_n\}$ and f on [0,1] such that

   $f_n(x) \to f(x)\ \forall x \in [0,1]$ but $\int_0^1 f_n(x)\,dx$ does not converge to $\int_0^1 f(x)\,dx$.

4. Suppose $\{E_n\}$ is a sequence of Lebesgue measurable sets on $\mathbb{R}$ such that $E_n \supset E_{n+1}$ for every n.
(a) Give a concrete example to show that it is not always true that $\lim_{n\to\infty} m(E_n) = m\left(\bigcap E_n\right)$. Here m denotes Lebesgue measure.
(b) Add a reasonable condition which will guarantee that $\lim_{n\to\infty} m(E_n) = m\left(\bigcap E_n\right)$. You need not prove this result.

5. Let $L^2[0,1] = \left\{ f : [0,1] \to \mathbb{R} : \int_0^1 |f(x)|^2\,dx < \infty \right\}$, and for all $f, g \in L^2[0,1]$ define the inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$.

(a) Define the norm $\|\cdot\|_2$ on $L^2[0,1]$. What is a norm (i.e. define "norm")?
(b) How can we be sure the inner product defined above exists and is finite? You may quote a result.
(c) Define the functions f_0, f_1 on $[0,1]$ by $f_0(x) = 1, f_1(x) = x, \forall x \in [0,1]$. Find the projection of the function f_1 onto the subspace spanned by f_0.