Analysis Qualifying Exam
August 24, 2007

Directions: Do as many problems as you can from each part—complex analysis and real analysis. To pass this exam, you must pass both complex analysis and real analysis. Justify all answers.

Complex analysis

1. Find all entire functions f with this property: there exists a constant C such that $|f(z)| \le C|z|$ for all $z \in \mathbb{C}$.

2. Evaluate $\int_{-\infty}^{\infty} \frac{\exp(ix)}{1+x^4} dx$.

3. (a) State Rouche's Theorem for holomorphic functions. Include the argument principle (i.e. the formula for the number of zeros of a holomorphic function in a suitably bounded region.)

(b) Use Rouche's Theorem to prove the Fundamental Theorem of Algebra.

4. Suppose that f is a holomorphic function in the open unit disk such that $|f(z)| \le f(1/2)$ for all z in the open unit disk. What can we say about f?

5. Let $f(z) = \frac{1}{\left(2(z+1/2)^2+1\right)\left((z-1/2)^2+1\right)}$. If we expand $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in a power series about 0, for which z does the series converge? Even if you are unable to give an exact answer, be as specific as you can..

Real analysis Note: all functions are real valued; all vector spaces are over the real numbers.

1. Suppose X is a normed linear space with norm $\|\cdot\|$.
(a) Define precisely: X is a complete normed linear space.
(b) Show that if X is a complete normed linear space, then every absolutely convergent series converges; i.e. that whenever $\sum_{n=1}^{\infty} x_n$ is a series in X such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then $\sum_{n=1}^{\infty} x_n$ converges to some element of X.
(c) Show that if X is a normed linear space such that every absolutely convergent series converges, then X is complete.

2. (a) Show that if $f \in L^1[0,1] \cap L^2[0,1]$ then $\|f\|_1 \le \|f\|_2$.
(b) Use (a) to show that $L^2[0,1] \subset L^1[0,1]$.

3. Consider the Hilbert space $L^2[0,1]$, let V be the linear subspace consisting of polynomials of degree less than or equal to one, and define the function f by $f(x) = x^2, 0 \le x \le 1$. Find the orthogonal projection of f onto V.

4. Consider the space $C[0,1]$ equipped with the maximum norm $\|f\| = \max_{x \in [0,1]} |f(x)|$.
(a) Prove or disprove: $S = \{f \in [0,1] : \|f\| \le 1\}$ is a compact subset of $C[0,1]$.
(b) Prove or disprove: the set $T = \{f \in C[0,1] : f, f' \in S\}$ has compact closure in $C[0,1]$.

5. Suppose f is an integrable function on $\mathbb{R}$. Define F by $F(x) = \int_{-\infty}^{x} f(t)\,dt$, so that F is the indefinite integral of f. Prove CAREFULLY that F is continuous.

6. Suppose $\{E_n\}_{n=1}^{\infty}$ is a sequence of Lebesgue measurable sets in $\mathbb{R}$. Denote Lebesgue measure by m.
(a) If the sequence is monotone increasing $(E_n \subset E_{n+1}\ \forall n)$ is it necessarily true that $m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$? If not true, give an example to show it need not be true, and add a simple condition which makes it true.
(b) If the sequence is monotone decreasing $(E_n \supset E_{n+1}\ \forall n)$ is it necessarily true that $m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$? If not true, give an example to show it need not be true, and add a simple condition which makes it true.

7. Suppose X is a normed linear space. Define what a bounded linear functional on X is.

(a) Show that if $x, y \in X$ with $F(x) = F(y)$ for all bounded linear functionals F, then $x = y$. Justify your argument by careful reference to relevant major theorem(s).

(b) Give a concrete representation of the bounded linear functionals on $L^p[0,1]$. Include any needed restrictions on p.