Analysis Qualifying Exam

Jan. 16, 2007

Part A Complex Analysis

(10pts) 1. Is it possible that an analytic function $f(z)$ has its real part given by
$$u(z) = x^2 - x - y \ , \quad z = x + iy \ ?$$

(10pts) 2. Compute $\int_P e^{-z^2} dz$, where $P = \{x+iy \mid y=1, x \in (-\infty, \infty)\}$ is oriented by increasing x.

Hint: $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

(10pts) 3. Suppose $f(z)$ is holomorphic on the entire z-plane and satisfies
$$|f(z)| \le A + B|z|^k \ , \quad \forall z \in \mathbb{C},$$
where A, B and k are positive constants. Prove that $f(z)$ is a polynomial of z.

(10pts) 4. Find a conformal mapping that carries the strip $\{x+iy \mid -\frac{\pi}{2} < y < \frac{\pi}{2}, x \in (-\infty, \infty)\}$ onto the unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$.

(16 pts) 5. Let D be the unit disk $\{z \in \mathbb{C} \mid |z|<1\}$ and Ω be an open domain containing $\overline{D}$. Suppose f is holomorphic on Ω with $|f(z)|=1$ for $|z|=1$.

(i) Prove that if $f \not\equiv$ constant, f must have a zero point in D.

(ii) Assume 0 is the only zero point of f in D and $f'(0) \neq 0$. Find all such $f(z)$ — prove your assertions.

Part B Real & Functional Analysis.

(10 pts) 6. Find $\displaystyle\lim_{k\to\infty} \int_0^{2007} f^k(x)\,dx$, where k are positive integers and f is a Lebesgue measurable function defined on $(0, 2007)$ with $|f(x)|<1$, $x \in (0, 2007)$.

(14 pts) 7. Let $\{f_n(x)\}_{n=1}^{\infty}$ be a sequence of Lebesgue measurable functions defined on $[0,1]$ with $0 \le f_n(x) \le 1$, $\forall n \ge 1$, $\forall x \in [0,1]$. What is the logical relationship among the following statements? Prove your assertions!

(a) $f_n(x) \to 0$ in measure on $(0,1)$ as $n \to \infty$.

(b) $f_n(x) \to 0$ in $L^1(0,1)$ as $n \to \infty$.

(10pts) 8. Let $f(x) = \dfrac{e^{-x} x^{\alpha+1}}{1-e^{-x}}$, $x \in (0, \infty)$, where $\alpha > -1$ is a constant.

(i) Show that $f \in L^1(0, \infty)$

(ii) Show that

$$\int_0^\infty f(x)\,dx = \Gamma(\alpha) \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+2}}, \text{ where}$$

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha+1}\,dt$$

(10pts) 9. Let $E \in [0, 1]$ be Lebesgue measurable. Show that $\forall \varepsilon > 0$, $\exists$ finitely open intervals $I_1, \cdots, I_n$ such that

$$m\left(E \,\Delta\, \bigcup_{k=1}^{n} I_k\right) < \varepsilon.$$

(10pts) 10. Let $\mathcal{B}$ be the set of Borel sets in $\mathbb{R}$, $\mathcal{M}$ be the set of Lebesgue measurable sets in $\mathbb{R}$. What is the logical relation between $\mathcal{B}$ and $\mathcal{M}$? Given a Lebesgue measurable set, how does it differ from Borel sets?

<u>No proofs required</u> !

(10pts) 11. Suppose $f(x)$ and $g(x)$ are positive Lebesgue measurable functions defined on $(0,1)$, satisfying

$$f(x)\,g(x) \geq 1 \quad , \quad \forall x \in (0,1).$$

Prove ~~that~~ $\left(\int_0^1 f(x)\,dx\right)\left(\int_0^1 g(x)\,dx\right) \geq 1$

(10pts) 12. Suppose $\infty > p > 0$, $\{f_k\}_{k=1}^{\infty} \subset L^p(0,1)$ satisfies

$$\|f_k\|_{L^p(0,1)} \to \|f\|_{L^p(0,1)}, \quad f_k \to f \text{ a.e on } (0,1),$$

as $k \to \infty$. Prove that $f_k \to f$ in $L^p(0,1)$ as $k \to \infty$.

(10pts) 13. Suppose $1 < p < \infty$. $\forall f \in L^p(0,\infty)$, define

$$(Tf)(x) = \frac{1}{x}\int_0^x f(y)\,dy, \quad \forall x > 0.$$

Prove Hardy's inequality:

$$\|Tf\|_{L^p(0,\infty)} \leq \frac{p}{p-1}\|f\|_{L^p(0,\infty)}.$$

(10pts) 14. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be orthonormal subsets of a Hilbert space H. If $\{x_n\}_{n=1}^{\infty}$ is a basis for H, and if

$$\sum_{n=1}^{\infty}\|x_n - y_n\|^2 < 1,$$

prove that $\{y_n\}_{n=1}^{\infty}$ is also a basis for H.

(10pts) 15. Let $\widetilde{H}_0^1 = \{ f \in AC[-2,2] \mid f(-2) = 0 = f(2),\ f \equiv \text{const on } [-1,1],\ f' \in L^2(-2,2) \}$

Prove that $\widetilde{H}_0^1$ is a Hilbert space under the inner product $(f,g) = \int_{-2}^{2} f'(x)\,g'(x)\,dx$

PART C.(all functions real valued and all scalars real)

11. Define $L^p[0,1]$ for each p, $1 \le p \le \infty$. State the Riesz Representation Theorem in this context.

12. Consider the Hilbert space $L^2[0,1]$.

a) Define the notion of orthonormal base.

b) Give an example of an orthonormal base. (You need not prove this.)

c) Sketch a proof of the fact that $L^2[0,1]$ and ℓ_2 (the space of square summable sequences) are isomorphic as Hilbert spaces.

13. Characterize with justification all continuous functions f on $[-1,1]$ such that

$\int_{-1}^{1} f(x)x^n dx = 0$ for all even integers $n = 0, 2, 4, \ldots$