Analysis Written Examination

August 25, 2008

This exam consists of two parts—complex analysis and real analysis. You must pass both parts to pass the exam.

Complex Analysis

1. Evaluate these integrals on the unit circle C with counterclockwise orientation.

   a) $\oint_C \exp\left(1+z^2\right) dz$

   b) $\oint_C \dfrac{\exp\left(1+z^2\right)}{z} dz$

   c) $\displaystyle\int_{-\infty}^{\infty} \frac{\exp(ix)}{1+x^2} dx.$

2. Let k be a positive integer and let $f:\mathbb{C}\to\mathbb{C}$ be an entire function such that $\left|f(z)\right| \le C\left(1+|z|\right)^k$ for all $z\in\mathbb{C},$ for some constant $C.$ Show that f is a polynomial of degree no greater than $k.$

3. Suppose $f:\mathbb{C}\to\mathbb{C}$ is a doubly periodic entire function. That is, suppose f is entire and there exist two nonzero complex numbers ω_1, ω_2 such that (i) ω_1/ω_2 is not real and

   (ii) $f(z+\omega_1) = f(z+\omega_2) = f(z)$ for all $z\in\mathbb{C}.$ Show that f must be constant.

4. Let $f(z) = \dfrac{z^2+4}{(3z+2)(z-3)}.$ We have $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in a neighborhood of $z=0.$

   Give the radius of convergence of this power series.

Real Analysis

1. Suppose $\{e_n\}_{n=1}^{\infty}$ is a countable orthonormal set in the Hilbert space H.
   a) Define "orthonormal ."
   b) Show that $\sum_{n=1}^{\infty} \frac{1}{n} e_n$ exists in H. (Show the partial sums form a Cauchy sequence and explain why this does the trick.)

2. Suppose $\{E_n\}$ be a sequence of Borel subsets of the real line and let m denote Lebesgue measure.
   a) Prove or disprove: if $E_n \subset E_{n+1}\ \forall n$ then $m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$.
   b) Prove or disprove: if $E_n \supset E_{n+1}\ \forall n,$ then $m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$.

3. a) Let V be the space of continuous functions on $[0,1]$ which have continuous derivatives on $[0,1]$, together with the max norm defined by $\|f\|_{\max} = \max\{|f(x)| : x \in [0,1]\}$. Let $S \subset V$ be the subset of functions f such that f' exists on (0,1), f' can be extended to be continuous on all of $[0,1]$ and $\|f\|_{\max} + \|f'\|_{\max} \le 1$. Show that any sequence $\{f_n\}$ in S has a subsequence which converges in $V, \|\bullet\|_{\max}$. (You will need to quote a famous theorem.)
   b) Show by example that the conclusion in a) does not hold if we instead define S to be the set of functions with $\|f\|_{\max} \le 1$.

4. Suppose f is a nonnegative measurable function on the measure space $(\mathrm{X}, \mathrm{B}, \mu)$ and suppose $\int f(x)\, d\mu(x) < \infty$. Show f is supported on a $\sigma -$finite set; i.e. show there is a countable collection $\{E_n\}$ of sets of finite measure with $f(x) = 0$ for all $x \notin \bigcup E_n$.

5. Suppose f is Lebesgue integrable on $(0,\infty)$. Define $F(s)=\int_0^\infty \frac{f(t)}{2+\cos(s^2t^2)}dt$ for all $s>0$.

   Show that F is continuous.

6. Suppose f is a Lebesgue integrable function on $\mathbb{R}$ and suppose g is a continuous function with compact support on $\mathbb{R}$. (I.e. g is continuous and vanishes outside a finite interval.)

   Let $h(x)=\int_{-\infty}^{\infty} f(y)g(x-y)dy\ \forall x.$

   a) Show that h is bounded.
   b) Show that h is continuous.
   c) Use a famous theorem on double integrals to show that h is Lebesgue integrable.