Master's Exam in Applied Mathematics   Fall 1993
Do 8 problems.

1. State the rule for differentiating $\int_{G(t)} F(x,t)\,dx$, where t is real and $G(t) \subset \mathbf{R}^n$. Use it to derive the differential conservation laws of mass and momentum for an ideal fluid:

$$\rho_t + \nabla\cdot(\rho \mathrm{v}) = 0$$
$$\rho \mathrm{v}_t + \rho(\mathrm{v}\cdot\nabla)\mathrm{v} = -\nabla p$$

2. Assume velocity and density are near an equilibrium state $\mathrm{v} = 0$, $\rho = \rho_0$ where ρ_0 is constant. Assume the pressure p is a function of density. Show that the pressure perturbation satisfies the wave equation (formally, to first order).
3. What is meant by the method of "descent" for the wave equation? Be precise but brief. What is the principle of "sharp cut-off"?
4. Explain what is meant by the Cauchy problem for the first-order pde $xu_x + 2yu_y = 0$ What are the characteristic curves? What do they have to do with existence and uniqueness of solutions to the Cauchy problem?
5. Let δ be the Dirac delta situated at $x = 0$ on the real line. Explain the equation $x\delta(x) = 0$. Express

$$f = \begin{cases} 1 & x < 1 \\ 2 & 1 \le x \le 3 \\ 1 & x > 3 \end{cases}$$

in terms of Heaviside function. Express f' in tems of Dirac deltas.
6. Prove the identity

$$\int_{\partial\Omega} uv_n - vu_n = \int_{\Omega} u\Delta v - v\Delta u$$

Use it to show the symmetry of Green's function $G(x,y)$ for Ω.
7. What is the Euler equation for extremizing the integral $\int_G |\nabla u|^2$ among functions with given boundary value? Assuming u satisfies this Euler equation, show that u **minimizes** the integral.
8. Consider heat flow in a bounded domain Ω. If the boundary is insulated, what is the appropriate boundary condition on the temperature? Write down formal solution in terms of an appropriate eigenfunction expansion. Explain meaning of all notation used. After a long time the temperature approches

a constant. About how long does this take, and what is the constant? Hint: Answer is in terms of eigenvalues.

9. Use Fourier transform to derive the formula for the fundamental solution of $u_t = u_{xx}$, i.e. the solution which is initially a delta function at origin.
10. Formally derive heat equation in one dimension from a symmetric random walk model.
11. Find the first term in the asymptotic expansion of

$$\int_0^\infty \exp(-\lambda z) z^2 g(z)\, dz$$

as $\lambda \to \infty$. Assume g analytic and $g(0) \neq 0$.

12. If ϵ is small, the solution to

$$y'' + \epsilon y = 0$$

satisfying $y(0) = 0$ and $y(1) = 1$ is approximately x. Can you improve on this?