

Do any 12

Qualifying Examination in Applied Mathematics 8-23-96

1. Solve the p.d.e.  $u_x + 6u_y = u$  subject to  $u = f(x)$  on the initial line  $y = 3x + 1$ . Give an example of a *characteristic* initial condition for this same p.d.e.
2. Explain *steepening of profile* relative to the evolution equation  $u_t + uu_x = 0$ . (The *profile* is the value of  $u$  at  $t = 0$ .) Give an example of some data for which eventually a classical solution will not exist. What does happen thereafter?
3. Use generalized functions to differentiate
$$\begin{cases} x & x < a \\ x^2 & x \geq a \end{cases}$$
4. For what values of  $a$  does the derivative exist in the ordinary sense?
  4. Do one of the following:
    - Define Green function for Laplace's equation for a bounded domain in  $\mathbb{R}^n$ . Prove that it is symmetric in its two variables.
    - Use method of images to find Green function for upper half plane.
  5. Derive from physical principles one of the following: wave equation, heat equation, continuity equation (= conservation of mass for fluid motion). Define all terms used in your derivation.
  6. Show that among all "competitors"  $u$  with given values on the boundary of a domain  $\Omega$  the energy  $\int_{\Omega} |\nabla u|^2$  is smallest when  $\Delta u = 0$ . What is the Euler equation for  $\int_{\Omega} p |\nabla u|^2$  where  $p$  is a smooth function on  $\Omega$ ?
  7. State some versions of the maximum principle for
    - Laplace's equation
    - Heat equation.
  8. Use multiple time scales to estimate the effect  $\epsilon > 0$  (small) has on the period of oscillation of  $\theta'' + (\theta - \epsilon\theta^3) = 0$ . You can assume  $\theta(0) = 1$  and  $\theta'(0) = 0$ .
  9. Define *asymptotic expansion*. Can two different function have the same asymptotic expansion?
  10. Consider the boundary-value problem  $\epsilon y'' + (1+x)y' + y = 0$ ,  $y(0) = y(1) = 1$ . Suppose  $\epsilon > 0$  is near 0. Show that a boundary layer forms, and state at which end it forms. Find the approximating solutions within the boundary layer, and outside the boundary layer. Match if you have time.
  11. What is the WKB approximation to  $\epsilon^2 y'' - q(x)y = 0$  where  $\epsilon \downarrow 0$ ?
  12. Let  $L$  be a linear differential operator with constant coefficients (on real line) and consider the evolution equation  $u_{tt} = Lu$ . Seek solutions of form  $\exp(i(kx - \omega t))$ . In this context what does *dispersive* mean? Explain difference between phase velocity and group velocity. (You can specialize to a particular example if that is easier.)

13. Contrast domain of dependence for wave eqn. in  $\mathbb{R}^1, \mathbb{R}^2, \mathbb{R}^3$ .