Qualifying Exam in Applied Mathematics January 1999

1. Find a good approximation for small $\epsilon > 0$ to the solution of

$$\epsilon y'' + y' = .5 \quad 0 < x < 1$$

subject to $y(0) = 0$, $y(1) = 1$.

2. Use multiple scales to find an approximation valid for large t (and small $\epsilon > 0$) to the solution of

$$u_{tt} + \epsilon u_t + u = 0$$

subject to the following conditions at $t = 0$: $u = 1$, $u_t = 0$. Give an example of a physical situation modeled by this problem.

3. Define *asymptotic expansion as* $t \downarrow 0$ relative to a sequence of functions $\{\phi_i\}$. What restrictions are there on the ϕ's?

4. What is the generalized derivative of

$$\begin{cases} x & x > 0 \\ 1 & x < 0 \end{cases}$$

5. Solve:

$$u_{xx} + u = \delta(x-1) \quad 0 < x < 2$$

subject to $u = 0$ at $x = 0$ and $x = 2$.

6. What is the *Rayleigh quotient* for an eigenvalue λ of the problem

$$-(pu_x)_x + qu - \lambda ru = 0 \quad a < x < b$$

with $u(a) = u(b) = 0$? What are its extremizing properties? Show that eigenfunctions u corresponding to distinct eigenvalues are orthogonal with respect to an appropriate weight function. What is the weight function? What conditions should the coefficients p, q, r satisfy?

7. State Fourier's law of heat conduction and use it to derive the heat equation (for the evolution of temperature in a conducting medium) in the plane or in 3-space. What boundary condition corresponds to an *insulated* boundary?

8. Solve $u_x = 3u_y$ subject to $u(x, 2x) = x$ for all real x. In this problem can $2x$ be replaced by ax for any real a? Explain your answer.

9. This problem concerns generalized (piecewise continuous) solutions of the nonlinear problem

$$u_t + (u^3)_x = 0$$

If u is continuous except at a moving interface, and is equal to 1 to the left of the interface and 2 to the right of the interface, *what is the speed of the interface?*

10. Solve

$$u_{xx} - u_{zz} = 1$$

$(x, z) \in R^2$ subject to $u = 0$ and $u_x = z$ on $x = 0$.

11. Define the *Green's function* for the Laplacian (domain corresponding to zero boundary conditions on the boundary of a nice region in 3-space). Explain how the solution to $\Delta u = f$ would be found in terms of the Green's function.