Applied Mathematics Qualifying Exam August 2002 Do 9 of these

(1) Using techniques of asymptotic analysis (i.e., *not* using the exact solution) find the first two terms in the asymptotic expansion for $\epsilon \downarrow 0$ of $y'' = -(1+\epsilon y)^{-2}$, $y(0) = 0$, $y'(0) = 1$.

(2)
- Define *stability* of an equilibrium point of the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$.
- Locate bifurcation points if any of the equation $y' = \mu - y(y-1)(y-2)$.

(3) Using techniques of asymptotic analysis (i.e., *not* using the exact solution) find a uniformly valid approximation for $\epsilon \downarrow 0$ to

$$\epsilon^2 y'' - y = -1$$

on $0 \le x \le 1$, where $y = 0$ at the boundaries.

(4) Using the technique of multiple scales (i.e., *not* using the exact solution) find the long time behavior of

$$y'' + \epsilon y' + y = 0, \quad y(0) = 1 \quad y'(0) = 0$$

for $\epsilon \downarrow 0$.

(5)
- Use characteristics to solve the Cauchy problem

$$u_x + x u_y = 1$$

subject to $u(x, x) = x$ for all x.
- If a and b are smooth functions then the Cauchy problem $a(x,y)u_x + b(x,y)u_y = 0$, $u(x,x) = x$ has a smooth solution in some neighborhood of any point on the line $y = x$. Is this TRUE or FALSE? (Explain your answer.)

(6)
- Derive the heat equation $u_t = \kappa \Delta u$ for the evolution of temperature u in a conducting medium.
- Make a change of variables which removes κ from the equation.
- Use the eigenfunction expansion of the solution with Neumann ($u_x = 0$) boundary conditions on $0 < x < 1$ to show that as $t \to \infty$ the solution approaches the average of the value at $t = 0$.

(7)
- Let v be the solution to

$$\begin{cases} \Delta v + v = \delta_{\mathbf{y}} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is some nice domain in $\mathbf{R}^n$, and $\delta_{\mathbf{y}}$ is the Dirac delta situated at the point $\mathbf{y} \in \Omega$. Let u be the solution to

$$\begin{cases} \Delta u + u = f & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

where f is bounded and continuous on Ω. Derive an expression for u in terms of v and f.
- Let w be the harmonic function in the unit disc $x^2 + y^2 < 1$ which has boundary value $x^4 + y^4$. Find the maximum value of u in the closed disc.

→(8)
- Suppose $u_{tt} = u_{xx}$ for $0 < x < 1$ and $t > 0$; at $t = 0$, $u = x^2$, $u_t = 1 + x^2$; $u(t, 0) = u(t, 1) = 0$ for all $t > 0$. Use an energy argument to find an upper bound for $\int_0^1 u_t^2 \, dx$ for $0 < x < 1$ and all t.

- Suppose

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left((1+x^2)\frac{\partial u}{\partial x}\right)$$

for $0 < x < 1$ and $t > 0$; at $t = 0$, $u = x^2$, $u_t = 0$. At $x = 0$ and $x = 1$, $u = 0$. Explain how you would use an eigenfunction expansion to represent u, but don't try to find the eigenfunctions.

- (9) For smooth u satisfying $u(1) = u(2) = 0$, characterize the minimum value of

$$\frac{\int_1^2 (1+x^2)u_x^2\,dx}{\int_1^2 u^2\,dx}.$$

(10) How would you define the notion of a *continuous* (not necessarily smooth) solution u to

$$\begin{cases} x^2 u_{xx} + u_{yy} = 0 & \text{in } x^2+y^2<1 \\ u = 0 & \text{on } x^2+y^2=1 \end{cases} \quad ?$$

- (11) Find the generalized derivative of

$$\begin{cases} x & -\infty < x < 1 \\ 2 & x > 1 \end{cases}$$