Applied Mathematics/PDE Qualifying Exam August 2003

Do 4 of the shorter problems (1–6) and 6 of the longer ones (7–14)

(1) (a) Give an example of a first-order 1D autonomous ($dx/dt = f(x)$) initial-value problem with more than one solution or explain why you think this is impossible to do.

(b) Which equilibrium point of $dx/dt = x(x-1)(x-2)(x-3)(x-4)(x-5)$ is the *most stable* (i.e., deviations from equilibrium decay fastest)? Quantify.

(2) Define the Sobolev space $H^k(R^n)$ and state the corresponding Sobolev imbedding theorem.

(3) In three-dimensional space the wave equation $u_{tt} = 4\Delta u$ is to be solved with initial data $u_t = 0$ everywhere and $u = 0$ except in the spherical shell $1 < |\mathbf{x} - \mathbf{a}| < 5$, in which $u < 0$. At what future times can you be sure that the solution is zero at the point $\mathbf{a}$? Are there times at which you can be sure the solution is *nonzero* at $\mathbf{a}$?

(4) Find two different solutions to

$$\Delta u = \delta_{\mathbf{a}}(\mathbf{x}),$$

one of which is unbounded at infinity. Δ is the Laplace operator in 3-dimensional space and $\delta_{\mathbf{a}}$ is the Dirac delta situated at the point $\mathbf{a} = (1, 1, 0)$.

(5) The *flow map* T_t (t near 0) corresponding to an autonomous system of ODEs in n-dimensional space is the map which takes a point a to the (assumed unique) solution at time t having initial value a. List the most significant properties of flow maps. Under what circumstances do the flow maps of two different systems commute?

(6) In representing functions of one variable x by *asymptotic series* for $x \to \infty$ we use *scale functions* $\phi_1, \phi_2, \ldots$. What requirements must scale functions satisfy? What does it mean to say

$$f(x) \sim \sum_0^\infty a_n \phi_n(x) \quad x \to \infty \, ?$$

Find two terms of such an asymptotic expansion for

$$\int_x^\infty e^{-t^2}\, dt.$$

[Hint: integrate by parts.]

Longer problems

(7) Use boundary layer theory to approximate the solution to

$$\epsilon y'' + y' + y = 0 \quad y(0) = 0,\ y(1) = 1$$

for $\epsilon \downarrow 0$.

(8) The boundary of the region $x^2 + y^2 \leq 1$, $y \geq 0$ is kept at zero temperature. At a certain instant of time the temperature is measured at each interior point. What can you say about the length of time it takes for the temperature to decay significantly from those values? [This problem requires a knowledge of *eigenfunction expansion*, estimation of smallest eigenvalue, and the relation of the latter to the question posed.]

(9) State and prove the *Weak Maximum Principle* for the PDE $\Delta u = 1 + |\mathbf{x}|^2 + u^2$ in a bounded regular domain of n-dimensional space.

(10) Use an energy argument to show that the solution to

$$\begin{cases} u_{tt} = u_{xx} & 0 < x < 1 \\ u = 0 & x = 1 \\ u = 0 & x = 0 \\ u = f(x) & t = 0 \\ u_t = g(x) & t = 0 \end{cases}$$

is unique. [Energy is $E(t) = \int_0^1 (u_t^2 + u_x^2)\, dx$.]

(11) Consider the Cauchy problem $u_t + uu_x = 0$, $u = g(x)$ on $t = 0$. Give examples of smooth data g such that

- the classical solution exists for all $t > 0$
- the classical solution ceases to exist at some $T > 0$

(12) Solve $xu_x - yu_y = u^2$ subject to $u(x, x) = x$. Here the "initial curve" is $y = x$. Are there exceptional initial curves and Cauchy data such that there exists no corresponding solution?

(13) Solve $y'' + 2y = \delta_a(x)$ subject to $y(0) = y(1) = 0$. [δ_a is the Dirac delta situated at the point $a \in (0, 1)$.]

(14) Introducing the new scaled time $\tau = \epsilon t$ as well as the unscaled time t, find an approximation valid for small ϵ (up to large t) to the problem

$$\frac{d^2y}{dt^2} + y + \epsilon y^3 = 0$$

with $y(0) = 1$ and $dy/dt(0) = 0$.