Qualifying Exam in Applied Mathematics January 2007

Do at least 10 problems

(1) Suppose f is a smooth vectorfield in R^n and $||f(x) - f(y)|| \leq 2||x - y||$ for all x, y. Prove, in this case, that the solution to the initial-value problem $x' = f(x)$, $x(0) = a$, is unique.

(2) Given the system $x' = 2x - y$, $y' = x^2 + y^2$, $x(0) = y(0) = 1$, estimate $x(.02)$.

(3) Derive the continuity equation of fluid motion: $\rho_t + \nabla \cdot (\rho v) = 0$ where ρ is the fluid density per unit mass and v is the (vector) velocity field.

(4) Explain how analytic functions of one complex variable are associated with irrotational (define) incompressible (define) flow. Describe the flow corresponding to $f(z) = z$.

(5) State strong and weak maximum principle for $u_t = \Delta u - (1 + u^2)$ on the unit disk in R^2, $t \geq 0$. If at $t = 0$, $u = x + y$, and $u = 1$ on the boundary of the disk at all times > 0, find an upper bound for u valid at all times and everywhere on the disk.

(6) Write down a solution to $u_t = \Delta u$ in R^n which, as $t \downarrow 0$ approaches the Dirac delta function. [Extra credit - derive it.]

(7) Solve $u_t = 4u_x$ subject to $u = \sin(x)$ at $t = 0$.

(8) Consider $u_t + uu_x = 0$. Give an example of an initial value $u_0(x)$ giving rise to a solution defined for all $t > 0$. Given an example of another initial value $u_1(x)$ such that the (classical) solution is not defined past a certain time.

(9) Find the dominant term in the asymptotic expansion as $t \to \infty$:

$$\int_0^\infty e^{-tx} \sin^4(2x)\, dx$$

(10) Explain $3\delta(3x) = \delta(x)$, where δ is the Dirac delta.

(11) We seek a harmonic function in the unit square, which equals x^2 on the base, $1 + x^2$ on the top, y^2 on the left side, and $1 + y^2$ on the right side. The solution is not $x^2 + y^2$ because that, while satisfying the boundary conditions, is not harmonic. State Dirichlet principle in general and use it to estimate $\int |\nabla u|^2$ here.

(12) Define Green's function G for the problem $\Delta u = f$ in a region and $u = 0$ on the boundary. Show $u = \int Gf$.

(13) Suppose u satisfies the wave equation $u_{tt} = \Delta u$ in a region and $\partial u/\partial n = 0$ on the boundary. Define the energy of a solution and show it is constant in time.

(14) (Continuation). Use the above to prove uniqueness of u given initial values of u and u_t.

(15) Let $\epsilon \downarrow 0$. Use boundary layer method (not the formula for the exact solution) to provide an approximation to the solution to $\epsilon u_{xx} + u_x = 2$, $0 < x < 1$ with $u(0) = 0$ and $u(1) = 1$. [Hint: There is a boundary layer of thickness $\sim \epsilon$ at the left boundary.

(16) Discuss method of spherical means for 3D wave equation.

(17) (Continuation) Describe method of descent to find solution in 2D.