Probability Preliminary Exam Fall 1995

1. Let $X : [0, 1) \rightarrow \mathbf{R}$ be the random variable on the triple $([0, 1), \mathcal{B}, dx)$, where $\mathcal{B}$ denotes the Borel field and dx denotes Lebesgue measure, given by $X(\omega) = \omega^2$. Let $Y(\omega) = I_{[1/2, 1)}(\omega)$ and let $\mathcal{F}$ denote the σ-algebra generated by Y. Find $E(X \mid \mathcal{F})$.

2. a. Give an example of a sequence of random variables $\{X_n\}$ which converges in probability to a r.v. Y, but which does not converge a.s. to Y.

b. Suppose that $P\{|X_n - Y| \geq \varepsilon\} \leq \varepsilon^{-1}(1 - 1/n)^{n^2}$ for each $\varepsilon > 0$. Does X_n converge to Y in probability, almost surely, or in distribution?

3. State and prove the central limit theorem for i.i.d. random variables.

4. Let $X_1, X_2, \ldots$ be a Markov chain in the countable state space S. Say what it means for the chain to be ergodic, and describe the stationary distribution in terms of return times.

Do one of the following three problems.

5. Suppose that $X_1, X_2, \ldots$ is an i.i.d. sequence of integrable random variables such that $E(\exp(tX_1)) < \infty$ for $t \in \mathbf{R}$. Prove the strong law of large numbers for the sequence.

6. Let $X_1, X_2, \ldots$ be a Markov chain of random variables in $\{0, \ldots, N\}$ with $X_1 = i_0$ and

$$P\{X_n = i \mid X_{n-1} = j\} = \binom{N}{i} \left(\frac{j}{N}\right)^i \left(\frac{N-j}{N}\right)^{N-i}, \quad i = 0, \ldots, N,$$

for $n > 1$. Find the probability that $\{X_n\}$ is eventually 0.

7. Let $X_1, X_2, ..$ be an i.i.d. sequence of $N(0, 1)$ random variables, and let $S_n = X_1 + \ldots + X_n$. Find $P\{\max_{2^n < k \leq 2^{n+1}} \frac{S_k}{\sqrt{2k \log(k)}} \geq 1 \text{ i.o.}\}$ by using the appropriate Gaussian tail estimate.