Probability and Statistics Qualifying Exam, August 2008

Please show all work for partial credit. If additional space is needed, attach pages to the back of the exam.

1. Suppose that the random variables (X, Y) are jointly uniformly distributed over the region defined by $0 \leq y \leq 1 - x^2$ and $-1 \leq x \leq 1$.

   (a) Find the marginal densities of $f(x)$ and $f(y)$.

   (a) Find the conditional densities $f(y|x)$ and $f(x|y)$.

2. An analysis of a text reveals that a vowel (v) is followed by a consonant (c) 80% of the time and a consonant is followed by a vowel 60% of the time.

   (a) If a letter is chosen at random, what is the probability that it is a vowel?

   (b) If three successive letters are chosen, what is the most likely sequence of vowels and consonants?

3. If X and Y are independent random variables, find $Var(XY)$ in terms of the means and variances of X and Y.

4. Let $X_1, X_2, \ldots, X_n$ be an i.i.d. sample from a Rayleigh distribution with parameter $\theta > 0$:

$$f(x|\theta) = \frac{x}{\theta^2} e^{-x^2/2\theta^2}, \qquad x \geq 0.$$

(a) Find the method of moments estimate of θ.

(b) Find the maximum likelihood estimate of θ.

(c) Find a sufficient statistic for θ.

5. A coin is thrown independently 10 times to test the hypothesis that the probability of heads is 0.5 versus the alternative that the probability is not 0.5. The test rejects the null hypothesis if either 0 or 10 heads are observed.

   (a) What is the significance level of the test?

   (b) If, in fact, the probability of heads is 0.1, what is the power of the test?

6. An elevator containing five people can stop at any of seven floors. What is the probability that no two people get off at the same floor? Assume the the occupants act independently and that all floors are equally likely for each occupant.

7. Suppose $\varepsilon_1, \ldots, \varepsilon_N$ are independent and identically distributed random variables following a normal distribution with mean 0 and variance 1. Suppose $Y_0 = 0$, and $Y_i = \theta Y_{i-1} + \varepsilon_i$, $i = 1, \ldots, N$ for $|\theta| < 1$. Find the maximum likelihood estimator of θ.

8. A chemist wishes to determine the percentages of impurities β_1 and β_2 in two 100 gram containers (1 and 2) of potassium chloride (KCL). The process she uses is able to measure the weight in grams of the impurities in any 2 gram sample of KCL with mean equal to the true weight of the impurities and standard deviation 0.006 gram. She makes three measurements. The first measurement is on a 2 gram sample from container 1, the second measurement is on a 2 gram sample from container 2, and the third measurement is on a mixture of a 1 gram sample from container 1 and a 1 gram sample from container 2.

   (a) Formulate this as a linear model.

   (b) Give formulas for unbiased estimators $\hat{\beta}_1$ and $\hat{\beta}_2$ of β_1 and β_2.

   (c) Estimate β_1 and β_2 for the three measurements 0.036, 0.056, and 0.058. Also calculate s^2, the unbiased estimate of σ^2, and compare it to the known variance of $(0.006)^2$.

9. Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with the density function $f(x|\theta) = \theta e^{-\theta x}$. Derive a likelihood ratio test of $H_o : \theta = \theta_o$ versus $H_a : \theta \neq \theta_o$, and show that the rejection region is of the form $\{\bar{X} \exp[-\theta_o \bar{X}] \leq c\}$.

10. A binary variable fluctuates between the values 0 and 1 according to the following discrete time transition matrix: $\begin{vmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{vmatrix}$ for $\alpha, \beta \in (0, 1)$.

    (a) Find the probability that a random variable that begins at 0 remains at 0 after two time steps.

    (b) Find the stationary distribution of this Markov chain.

    (c) For what values of α and β is this chain reversible?