Scientific Computation Qualifying Exam
September, 2002

1. Let A be an invertible $n \times n$ matrix and let $\vec{u}, \vec{v} \in \mathbb{R}^n$.
   (a) What is the rank of $\vec{u}\vec{v}^T$?
   (b) Prove the Sherman-Morrison formula:
   $$(A + \vec{u}\vec{v}^T)^{-1} = A^{-1} - (A^{-1}\vec{u}\vec{v}^T A^{-1})/(1 + \vec{v}^T A^{-1}\vec{u})$$
   (c) Explain how you would exploit the Sherman-Morrison formula to solve $B\vec{x} = \vec{c}$ where B is of the form
   $$B = \begin{pmatrix} * & & & & \alpha \\ * & * & & & \\ * & * & * & & \\ \vdots & \vdots & \vdots & \ddots & \\ * & * & * & \cdots & * \end{pmatrix}$$

2. Given $f(x) = x - \ln(x+1) - 7$
   (a) Prove that $f(x)$ has only one positive root.
   (b) Determine a function $g(x)$ and an interval $[a, b]$ on which the fixed-point iteration $x_{k+1} = g(x_k)$ converges to a positive solution of $f(x) = 0$. Demonstrate the convergence.
   (c) Once x_k is 'near' this root, about how many decimal places of accuracy do you expect to gain per iteration?

3. An $n \times n$ matrix A is tridiagonal if $A_{ij} = 0$ for $|i - j| > 1$.
   (a) How many multiplications/divisions are required to determine the LU factorization of a tridiagonal matrix A, assuming no pivoting is required? How many multiplications/divisions are required to solve $LU\vec{x} = \vec{b}$ by forward and backward substitution?

(b) Alternatively, one could solve $A\vec{x} = \vec{b}$, A tridiagonal, by first computing the matrix A^{-1} and multiplying $\vec{x} = A^{-1}\vec{b}$. How many multiplications/divisions are required to solve $A\vec{x} = \vec{b}$ using this alternative? How many multiplications/divisions would be required if A^{-1} was already available?

(c) Now consider the problem of solving a system $A\vec{x} = \vec{b}$ where A is a banded matrix of bandwidth m: $A_{ij} = 0$ for $|i-j| > m$. (When $m = 0$, A is diagonal; when $m = 1$, A is tridiagonal.) Assuming no pivoting, discuss storage requirements and the number of multiplications/divisions for Gaussian elimination and forward and backward substitution.

4. We wish to fit the data points (0,1), (1,3), (4,5) to a function of the form:

$$f(x) = ax + b\sqrt{x}$$

(a) Which of the following choices for the parameters a, b gives a better fit: $a = 1, b = 2$ or $a = 2, b = 1$? Justify your answer.

(b) Is either of these the least squares solution? Justify your answer.

5. For a symmetric $n \times n$ matrix A, discuss two algorithms you could implement to decide if A is positive definite. What is the leading order operation count required for each algorithm?

6. Suppose we solve the equation $y'(t) = f(y)$ using the multi-step method

$$Ay_n - 4y_{n-1} + By_{n-2} = 2\Delta t f(y_n)$$

(a) Find the constants A and B for the method to be consistent with the ODE.

(b) What is the local truncation error of this method?

7. Consider the equation $u_t = au_x$ where $0 < a$, initial conditions $u(x,0) = f(x)$ and $-\infty < x < \infty$. Suppose we use the numerical method

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{a}{12h}\left[u_{j-2}^n - 8u_{j-1}^n + 8u_{j+1}^n - u_{j+2}^n\right]$$

(a) Describe the numerical domain of dependence of u_j^n and the domain of dependence of the exact solution $u(jh, n\Delta t)$.

(b) What condition on Δt and h is necessary for the numerical domain of dependence to include the domain of dependence of the exact solution?

(c) Find the amplification factor of this method and determine the CFL condition.

8. Consider the boundary-value problem

$$u''(x) + bu'(x) + ku(x) = f(x), \qquad u(0) = \alpha,\ u(1) = \beta$$

where $b > 0$ and $k > 0$ are constants.

(a) Discretize the equation using second-order difference operators D_0 and D_+D_-.

(b) Let $x_0 = 0$, $x_{n+1} = 1$ and $x_j = jh$ where $h = \frac{1}{n+1}$. Let u_j represent the numerical solution at x_j for $j = 1, 2, \ldots, n$. Write your numerical method as a linear system $A\vec{U} = \vec{F}$ where $\vec{U} = (u_1, u_2, \ldots, u_n)^T$. Write clearly and explicitly the entries of A and $\vec{F}$.

(c) Define *stability* for this method.

9. Suppose we solve Poisson's equation in the unit square $(x, y) \in [0, 1] \times [0, 1]$ with Dirichlet boundary conditions using the numerical method

$$\frac{(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + (u_{i,j+1} - 2u_{i,j} + u_{i,j-1})}{h^2} = f_{i,j}, \quad \text{for } 1 \le i, j \le n$$

$$u_{i,j} = g_{i,j} \quad \text{for } (i, j) \text{ on the boundary}$$

where $h = \frac{1}{n+1}$ is the grid spacing in both directions.

(a) Prove that when $f_{i,j} = 0$ for all i, j, the maximum and minimum values of $u_{i,j}$ are attained at the boundary.

(b) Suppose we write the numerical method as a linear system $A\vec{u} = \vec{F}$ where A is $n^2 \times n^2$. Comment on the structure of the matrix A and use the result from part (a) to determine if this system has a unique solution for arbitrary $f_{i,j}$.

10. Consider the heat equation $u_t = u_{xx}$ with Dirichlet boundary conditions $u(0,t) = u(1,t) = 0$. Suppose we approximate the solution with the implicit method

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2}, \qquad u_0^n = u_{m+1}^n = 0$$

where $h = \frac{1}{m+1}$.

(a) Find the local truncation error.

(b) Suppose that to advance one step from n to $n+1$ we use the iterations

$$u_j^{n,k+1} = u_j^n + \frac{\Delta t}{h^2}\left(u_{j+1}^{n,k} - 2u_j^{n,k+1} + u_{j-1}^{n,k}\right), \quad u_j^{n,0} = u_j^n$$

Define $\delta_j^k = u_j^{n,k} - u_j^{n+1}$ and find the iteration equation for δ_j^k. Derive the condition on Δt and h for this iteration to converge as $k \to \infty$.