Scientific Computation Qualifying Exam
January, 2003

1. Let $\vec{v} \in \mathbb{R}^n$.

    (a) What is the rank of $\vec{v}\vec{v}^T$?

    (b) Show that:
$$(I + \vec{v}\vec{v}^T)^{-1} = I - (\vec{v}\vec{v}^T)/(1 + \vec{v}^T\vec{v})$$

    (c) Show that $I + \vec{v}\vec{v}^T$ is positive definite

    (d) What are the eigenvalues of $I + \vec{v}\vec{v}^T$?

    (e) Using the *conjugate gradient* algorithm to solve $(I + \vec{v}\vec{v}^T)\vec{x} = \vec{b}$, what is the maximum number of conjugate gradient iterations required (assuming exact arithmetic)? Explain.

2. Consider the following iteration: $x_0 = 0, \qquad x_{i+1} = \sqrt{2 + x_i}$

    (a) Prove that
$$\lim_{i \to \infty} x_i = 2$$

    (b) What is the order of convergence?

3. Consider $A\vec{x} = \vec{b}$, $A \in \mathbb{R}^{n\times n}, \vec{b} \in \mathbb{R}^n$ and $A(\vec{x} + \vec{\Delta x}) = \vec{b} + \vec{\Delta b}$.

    (a) Give a relation between the relative change in the solution compared to the relative change in the right-hand side.

    (b) How would you estimate the condition number of A?

4. Give an example of

    (a) a well-conditioned matrix

    (b) an ill-conditioned matrix

5. Consider the problem

$$\mathcal{L}y = y''(x) + y(x) = 0, \quad \text{for } 0 < x < 2\pi \qquad \text{with BC: } y(0) = y(2\pi) = 1,$$

and the discrete operator (for $h = 2\pi/(n+1)$ and $x_k = kh$, $k = 1, \ldots, n$)

$$\mathcal{L}^h y_k = \frac{y_{k-1} - 2y_k + y_{k+1}}{h^2} + y_k$$

(a) Find the exact solution of the ODE.

(b) Set $y_k = \cos(kh)$. Show that $|\mathcal{L}^h y_k| \le O(h^p)$. What is p?

6. Suppose you need to approximate the integral

$$\int_{-A}^{A} e^{-x^2}\,dx$$

to an accuracy of 10^{-4} using the Midpoint rule. If $h = 2A/n$, what value of n would you expect to have to use?

7. Consider the heat equation $u_t = \alpha u_{xx}$ where $0 < \alpha$, initial conditions $u(x,0) = f(x)$ and $-\infty < x < \infty$. Suppose we approximate the solution using the Crank-Nicolson method

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\alpha}{2}\left\{\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2} + \frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{h^2}\right\}$$

(a) Describe the numerical domain of dependence of u_j^n. Can the numerical domain of dependence include the domain of dependence of the exact solution?

(b) Find the amplification factor of this method and determine the stability condition.

8. Assume $f(x)$ is smooth and $0 < f(x) < 1$ for all $x \in \mathbb{R}$. Consider the problem

$$u_t - f(x)u_x = 0, \qquad -\infty < x < \infty \qquad u(x,0) = g(x)$$

Suppose we use the upwind method

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - f_j\frac{u_{j+1}^n - u_j^n}{h} = 0,$$

(a) Find the local truncation error.

(b) Use the amplification factor to determine a condition for stability.