Scientific Computation Qualifying Exam
June 2007

1. Choose a matrix norm and compute the condition number of
$$A = \begin{pmatrix} \epsilon & 1 \\ 1 & 1 \end{pmatrix}$$

2. Let A be an $n \times n$ symmetric, positive definite matrix. Explain the Cholesky factorization and what it can be used for.

3. Suppose A is an $n \times n$ symmetric square matrix and we use the conjugate gradient (CG) method to solve $Ax = b$. There is a theorem that says that if $A = I + B$ where $rank(B) = r \leq n$, then the conjugate gradient method converges in at most r steps.

   Consider first using CG to solve the problem where
$$A = \begin{pmatrix} 1000 & -1 & 0 \\ -1 & 100 & -1 \\ 0 & -1 & 10 \end{pmatrix}$$
   Then consider solving the equivalent but preconditioned problem $Mz = q$ where $M = C^{-1}AC^{-1}$, $z = Cx$ and $q = C^{-1}b$. Using
$$C = \begin{pmatrix} \sqrt{1000} & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & \sqrt{10} \end{pmatrix}$$
   find the matrix M. Does the new system gives advantages when using CG? Explain why using the given matrix C makes sense. Give convincing arguments.

4. (a) Prove the following statement:

   A vector $x \in \mathbb{C}^n$ minimizes $\|Ax - b\|_2^2$ for $A \in \mathbb{C}^{m \times n}$ and $b \in \mathbb{C}^m$ if and only if $Ax - b$ is orthogonal to the range of A.

   (b) Explain how this statement relates to the least squares solution of $Ax = b$.

   (c) Find the least squares solution of
$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \\ 1 & 5 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

5. Suppose $A = M - N$ where M is easy to invert. Explain how to solve $Ax = b$ using Jacobi iterations. Explain also the difference between using Gauss-Seidel instead of Jacobi.

6. Derive the open Newton-Cotes quadrature formula
$$\int_{x_0}^{x_3} f(z)dz = \frac{3h}{2}\Big[f(x_1) + f(x_2)\Big] + Ch^p f''(\xi)$$
   and find C and p.

7. Suppose you are given four points (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3), and you use a ***natural*** spline to approximate the function that generated the data.

   (a) How many cubic pieces will the spline have?

   (b) Suppose the cubic piece k is $S_k(x) = a_k + b_k(x - x_k) + c_k(x - x_k)^2 + d_k(x - x_k)^3$. Write down all the conditions that must be satisfied by the coefficients of all cubic pieces.

8. Suppose you approximate $f'(x)$ by $\frac{1}{2h}[-f(x + 2h) + 4f(x + h) - 3f(x)]$.

   (a) find the order of this approximation

   (b) use Richardson extrapolation to increase the order of the approximation and give the new formula.

9. Suppose you solve the equation $y'(x) = f(y)$ using the Runge-Kutta method

$$
\begin{aligned}
K_1 &= \frac{1}{3}hf(y_n) \\
y_{n+1} &= y_n + h\Big[af(y_n) + bf(y_n + K_1)\Big]
\end{aligned}
$$

   (a) What condition do you need to impose on the coefficients a and b for the method to be consistent with the ODE?

   (b) Find the coefficients for the method to have local truncation error $O(h^p)$ with p as large as possible. What is your value of p? Explain. Show the local truncation error using your coefficients.

10. Consider the one-dimensional problem for $0 \le x < 1$,

$$u''(x) + u'(x) - u(x) = f(x), \quad u(0) = 1, \quad u(1) = 0$$

   Use second order finite differences to discretize the ODE with $x_j = jh$ for $j = 0, 1, \ldots, M - 1$ and $h = 1/M$. Write this method as a matrix equation. What method would you use to solve the linear system and why?

11. Write down the Crank-Nicolson method for solving the heat equation $u_t = \mu(u_{xx} + u_{yy})$ in 2D in the unit square. Assume the boundary conditions are Dirichlet. Perform the von Neumann stability analysis, find the amplification factor and discuss the stability condition for this method.