Topology Written exam
January 19, 1992 3 ?

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DATE 1/9/93
test for J. Earl (Lawson)

1. a. Describe the construction of the fundamental group including the group operation and group properties
b. Describe the relation of π_1 to covering spaces.
c. Compute π_1 for the following spaces - give a rough sketch of the idea in each case.
(1) S^2
(2) S^1
(3) $S^1 \times S^1$
(4) $\mathbb{R}P^2$

2. Let K be a simplicial complex. Describe the construction of the simplicial chain complex for K. Indicate how one constructs simplicial homology $H_*(K)$ from this chain complex. There are certain properties (axioms) for which characterize simplicial homology - give them.

3. Compute the simplicial homology for the following spaces
(a) S^2
(b) $S^1 \times S^1$
(c) $\mathbb{R}P^2$
(d) S^n, $n > 2$,
(e) $S^2 \vee S^2$ (the 1 point union of 2 copies of S^2.)

4. Describe the Mayer-Vietoris sequence. By dividing $S^1 \times S^2$ into $S^1 \times S^2_+ \cup S^1 \times S^2_-$, where $S^2_\pm$ denote the upper and lower hemispheres use the Mayer-Vietoris sequence to compute the homology of $S^1 \times S^2$.

5. Let $p: \tilde{X} \to X$ be a covering map. Give necessary and sufficient conditions involving the fundamental group
(over)

for a map $f: Y \to X$ to lift to a map $\bar{f}: Y \to \tilde{X}$ so that $f = p\bar{f}$. If $f: Y \to X$ is itself a covering map, give conditions involving fundamental groups that imply $\bar{f}$ is a homeomorphism.

6. Let X be a compact Hausdorff space, Y a Hausdorff space. Suppose $f: X \to Y$ is continuous and maps onto Y. Define $u \sim v$ iff $f(u) = f(v)$. Show $X/\sim$ is homeomorphic to Y. Use this to show that if $X = D^2$ and $x \sim y$ iff $|x| = |y| = 1$ and $x = -y$, then $X/\sim$ is homeomorphic to $\mathbb{RP}^2 = S^2 / x \sim -x$. (Hint: Define an appropriate map $D^2 \to \mathbb{RP}^2$ so one can use the above proposition.)

7. A space is called countably compact if every covering by a countable number of open sets has a finite subcovering. It is called limit point compact if every infinite set has a limit point. Show that a Hausdorff space is countably compact iff it is limit point compact.

8. a) Prove that the product of two connected spaces is connected.

(b) Show that the product of two path connected spaces is path connected

(c) Discuss the relationship between the concepts of path connectedness and connectedness — e.g., are they equivalent?

(d) Show that an open subset of the plane is connected iff it is path connected.

9. Define a locally compact space. Let X be a locally compact space that is not compact. Define and discuss the one-point compactification of X. If $X = \mathbb{R}^2$, its one-point compactification is ~~homeomorphic~~ a familiar space. What is it? Justify your answer.

Executive