Topology Written Examination

Friday, August 26, 1994 – 9:00 A.M. - 1:00 P.M.

Gibson Hall, Room 415

Let X and Y be Hausdorff topological spaces.

1. a) Define: X is compact.

   b) Define: X is connected.

   c) Define: The product topology on $X \times Y$.

   d) Prove: If X and Y are compact, so is $X \times Y$.

   e) Prove: If X and Y are connected, so is $X \times Y$.

1. a) Define: X is locally compact.

b) Define: $A \subset Y$, the relative topology on A.

c) Let W be the set $X \cup \{\infty\}$, where ∞ is a point not in X. Describe the topology on W that makes W the "one-point compactification" of X.

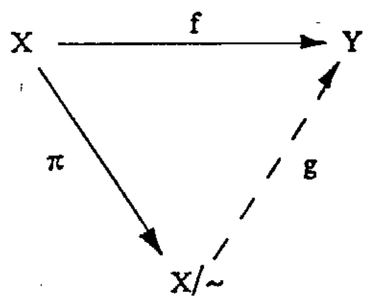
d) Prove: W is compact.

e) What is the relationship between the original topology on X and the subspace topology on X as a subset of W?

3. Let $\sim$ be an equivalence relation on X.

a) Define the quotient topology on $X/\sim$.

b) Let $f : X \to Y$ be a continuous function.



Under what conditions can we assert the existence of a continuous function g such that $f = g \circ \pi$?

c) Let $X = [0,1] \times [0,1]$, and let

$$f : [0,1] \times [0,1] \to S^1 \times S^1$$

be defined by

$$f(s,t) = \left(e^{2\pi i s}, e^{2\pi i t}\right) \quad , \quad 0 \leq s,t \leq 1$$

Discuss f and the quotient space it induces.

What about a function g?

4. Compute the fundamental group of the following spaces. In each case give details as indicated.

(a) the circle (sketch the argument using the covering space $\mathbb{R} \to S^1$)

(b) the projective plane $\mathbb{R}P^2$ (use the covering space $S^2 \to \mathbb{R}P^2$ and the fact that S^2 is simply connected).

(c) the product $\mathbb{R}P^2 \times S^1$ (use (a), (b))

(d) the Möbius band (use (a))

(e) the torus $S^1 \times S^1$.

5 Recall that a space is contractible if it is homotopically equivalent to a point.

(a) Show that if Y is contractible, then $X \times Y$ is homotopically equivalent to X.

(b) Show that if Y is contractible, then any two maps from X to Y are homotopic.

6. Compute the homology of each of the following spaces. You may use singular homology or cellular homology. We are assuming integral coefficients.

   (a) the n-sphere S^n

   (b) the real projective plane $\mathbb{R}P^2$

   (c) the Klein bottle K (use the CW decomposition coming from thinking of K as the quotient space of the rectangle with opposite edges identified)

   (d) the annulus A (region between two circles)

   (e) the relative homology of the pair $(A, \partial A)$, where ∂A denotes the two circles constituting the boundary of the annulus (<u>Hint</u>: Use excision to reduce from a simpler pair to this one, or use the long exact sequence of the pair.)

7. Use the Mayer-Vietoris sequence to compute the homology groups (Z -coefficients) of the torus $T^2 = S^1 \times S^1$.

8. Use the following argument to show every map $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point

(a) First show that f lifts to a map $\widetilde{f} : S^{2n} \to S^{2n}$ of the covering space S^{2n} of $\mathbb{R}P^{2n}$.

$$\begin{array}{ccc} S^{2n} & \xrightarrow{\widetilde{f}} & S^{2n} \\ \downarrow{\scriptstyle p} & & \downarrow \\ \mathbb{R}P^{2n} & \xrightarrow{f} & \mathbb{R}P^{2n} \end{array}$$

(b) Then show that any map $\widetilde{f} : S^{2n} \to S^{2n}$ must send some point x to itself or to its antipode $-x$.