Topology Exam, 1995

1. Show that a compact Hausdorff space is normal.

2. (A). Show that the closed interval $[0,1]$ is connected.

   (B). The cone $C(X)$ over a space X is formed from $X \times I$ ($I = [0,1]$) by identifying the points of the set $X \times \{1\}$ to a single point v, called the vertex of $C(X)$. Show that $C(X)$ is connected.

3. The *diameter* of a nonempty subset of a metric space (X, d) is defined to be

$$\text{diam}(E) = \sup\{d(x,y) \mid x, y \in E\} .$$

   Show that if $\{E_k\}_{k=1}^{\infty}$ is a decreasing sequence of closed nonempty subsets of a complete metric space whose diameters tend to zero, then $\bigcap_{k=1}^{\infty} E_k$ consists of precisely one point.

4. For each of the following spaces, give π_1 and H_* (Z-coefficients for H_*). Indicate how you are getting your answer – a full proof is not required.

   (a) S^1

   (b) S^n, $n > 1$

   (c) $\mathbb{R}P^2$ (real projective plane)

   (d) $S^1 \times \mathbb{R}P^2$

   (e) $S^1 \times S^2$

   (f) $\mathbb{R}^2 - \{(0,0) \cup (0,1)\}$

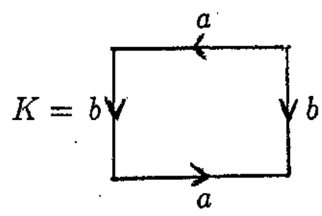
5. Describe how a short exact sequence of chain complexes leads to a long exact sequence in homology. Use your description to obtain:

   (a) long exact sequence for a pair (X, A)

   (b) the Mayer-Vietoris sequence for $X = U_1 \cup U_2$, where U_1 and U_2 are open.

   (c) indicate how to use (a), (b) to compute the homology groups of S^n.

6 (a) Give a CW decomposition of the Klein bottle



   (b) Use (a) to compute the homology of K with Z coefficients and then with Z_p coefficients p-odd prime.

   (c) Compute $H_*(S^1 \times K; Z_p)$.

7. Show that every map $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point.

   (Hint: use the fact that there is a 2-fold covering $\pi : S^{2n} \to \mathbb{R}P^{2n}$, and try to lift the map f to a map between S^{2n}).