Topology Written Examination
January 13, 1998

1. Show that a closed subset of a compact space is compact.

2. Define the one point compactification $X^+$ of a locally compact Hausdorff, including how the open sets are defined. Show that $X^+$ is compact.

3. Recall that if $X$ is a metric space, and $\mathcal{U}$ is an open cover, then $\delta$ is the Lebesgue number of the cover if every set of diameter less than $\delta$ is contained in an element of the cover. A theorem states that every compact metric space possesses a Lebesgue number. Use the concept of Lebesgue number to show that if $X$ is a compact metric space, $Y$ is a metric space, and $f : X \to Y$ is a continuous function, then $f$ is uniformly continuous, i.e., given $\epsilon > 0$, there exists $\delta > 0$ so that $d_X(x_1, x_2) < \delta$ implies $d_Y(f(x_1), f(x_2)) < \epsilon$.

4. Show that a path connected space is connected.

5. (a) Define what is meant by the quotient topology on a space $Y$ formed from the space $X$ by making certain identifications of points.

(b) Show that if $X$ is a compact space, $Y$ is a Hausdorff space, and $f : X \to Y$ is a surjective continuous map, then if we form an identification space $\overline{X}$ from $X$ where for each $y \in Y$, all points in $f^{-1}(y)$ are identified, then there is a homeomorphism from $\overline{X}$ to $Y$.

(c) Show that the the quotient space of the unit square $R = [0, 1] \times [0, 1]$ in the plane where $(0, y) \sim (1, y), (x, 1) \sim (x', 1), (x, 0) \sim (x', 0)$ is homeomorphic to the 2-sphere.

6. Suppose $X$ is a path connected space and $x_0 \in X$. Define the fundamental group $\pi_1(X, x_0)$, including the definition of the group operation and a sketch of why this forms a group. If $x_1$ is a different basepoint, sketch the argument that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

7.(a) Discuss how the covering map $\phi : \mathbb{R} \to S^1, \phi(t) = e^{2\pi i t}$ is used to show that the fundamental group of the circle in isomorphic to the integers, sketching the ideas of the proof.

(b) Use a similar idea and the covering map $S^2 \to \mathbb{RP}^2$ given by thinking of $\mathbb{RP}^2$ as a quotient space of $S^2$ given by identifying antipodal points to show that $\pi_1(\mathbb{RP}^2, x_0) \simeq \mathbb{Z}/2\mathbb{Z}$.

8. Compute the fundamental group and homology groups of each of the following spaces, giving a sketch in each case of how one computes the result.

(a) $S^1$

(b) $S^3$

(c) $S^1 \times S^1$

9. Describe what is meant by a finite CW complex. Explain how cellular homology (i.e. the homology which is computed using the CW complex structure) is computated, and the relation of cellular homology to singular homology. Use cellular homology to compute the homology of the following spaces, explaining the CW complex you use.

(a) $X$ is the quotient of the solid equilateral triangle, where the sides are identified by a rotation of $2\pi/3$ about the center point of the triangle. (Hint: $X$ should have one 0-cell, one 1-cell, and one 2-cell.)