Fall 98 Topology Written Exam

1. Show that if A is a connected subset of X and if $A \subset B \subset \overline{A}$, then B is connected. Give an example to show that this is not true if we replace connected by path connected in the above statement.

2. Show that a Hausdorff space is normal iff for any sets U open and C closed with $C \subset U$ there is an open set V with $C \subset V \subset \overline{V} \subset U$.

3. Prove that the product of two compact sets is compact.

4. State Urysohn's Lemma and the Tietze Extension Theorem. Show that the Tietze Extension Theorem implies Urysohn's Lemma.

5. Define what is meant by Y being a quotient space of X, including how the quotient topology on Y is defined. Suppose Y is a quotient space of X, with quotient map $p : X \to Y$, and $f : X \to Z$ is a continuous map, X is compact, Y is Hausdorff, and $f(x_1) = f(x_2)$ iff $p(x_1) = p(x_2)$. Show how to use this information to construct a homeomorphism from Y to Z. Use this method to construct a homeomorphism from the quotient space of $[0,1] \times [0,1]$ formed by identifying $(a, 0)$ to $(a, 1)$ and identifying $(0, b)$ to $(1, b)$ to the torus $S^1 \times S^1$.

6. Suppose that W is connected and $p : X \to Y$ is a covering map and $f : W \to Y$ is continuous. Let $\tilde{f}_1, \tilde{f}_2 : W \to X$ be liftings of f and assume $\tilde{f}_1(w_0) = \tilde{f}_2(w_0)$ for some $w_0 \in W$. Show that $\tilde{f}_1 = \tilde{f}_2$. This shows that liftings from a connected space are determined by their value at a point. Give a condition involving fundamental groups for a lifting to exist, given that W is path connected and locally path connected. Sketch how this condition is used to get the lifting.

7. Compute the fundamental group of each of the following spaces. Give a justification (one well constructed paragraph is sufficient) in each case.

(a) S^1

(b) $\mathbb{RP}^2$

(c) $S^1 \times \mathbb{RP}^2$

(d) the space formed from a pentagon by identifying the edges by rotation by $2\pi/5$.

(e) the CW complex $X = S^1 \cup_f e^2 \cup_g e^3$ where $f : S^1 \to S^1$ is $f(z) = z^4$ and $g : S^2 \to S^1 \cup_f e^2$ is the composition of projection of S^2 to D^2 which is then identified with the cell e^2 and the characteristic map from e^2 to $S^1 \cup_f e^2$.

8. Compute the homology of the following spaces:

(a) the Möbius band

(b) the space from 7(d).

(c)the space X from 7(e)

(d) the pair (X, S^1), where X is the space from 7(e) (Hint: Use the long exact sequence for the pair)

(e) the space $X = X_1 \cup_f X_2$ where $X_1, X_2 = S^1 \times D^2$ and $f : S^1 \times S^1 \subset X_2 \to S^1 \times S^1 \subset X_1$ is the map $f(z_1, z_2) = (z_1, z_1 z_2)$. (Hint: Use the Mayer-Vietoris sequence with the two halves

X_1, X_2 (saying why it applies here) and determine what the induced maps from the inclusions of the intersection - which you should regard as lying in X_2 and identified with points in X_1 through f.

9. Use homology to show that there is no retraction from D^3 to S^2. Use this to show that any continuous map $f : D^3 \to D^3$ must have a fixed point.