TOPOLOGY EXAM

August 28, 2000

1. Let $\mathbb{R}^n$ be n-dimensional Euclidean space with the usual topology. Let $X = \mathbb{R}^n \cup \{\infty\}$, where ∞ is a point not in $\mathbb{R}^n$.

   a) Describe the topology on X that makes X the one-point compactification of $\mathbb{R}^n$.

   b) Show that X is homeomorphic to S^n, the n-dimensional sphere.

2. A space X is locally path connected if for every point x of X and every neighborhood U of x, there is a path-connected neighborhood V of x contained in U.

a) Give an example of a space that is connected, but not path connected.

b) Show that a space X that is connected and locally path connected is path connected.

3. A space X is *limit point compact* if every infinite subset of X has a limit point. A space X is sequentially compact if every sequence in X has a convergent subsequence.

a) Show that every compact space X is limit point compact.

b) Show that every compact space X is sequentially compact.

4. Use the map $p : \mathbb{R} \to S^1$ given by $p(s) = (\cos 2\pi s, \sin 2\pi s)$ to calculate $\pi_1(S^1)$.

5. A space X is contractible if the identity map of X is homotopic to a constant map, i.e., the identity map is nullhomotopic.

a) Show that a space X is contractible iff every map $f : X \to Y$, for arbitrary Y, is nullhomotopic.

b) Show that X is contractible iff every map $f : Y \to X$ is nullhomotopic.

6. Compute the fundamental group and all homology groups of the following spaces:

(a) $\mathbb{P}^2$

(b) S^4

(c) $S^1 \times \mathbb{R}^2$

(d) $S^1 \vee S^1$

7. Define "finite CW complex." Explain how to compute its cellular homology. Use cellular homology to compute the homology of the following spaces, describing the CW complex you use.

a) S^1

b) S^{10}

c) $\mathbb{CP}^2$

d) $S^1 \cup_f e^2$, where $f : S^1 \to S^1$ is $f(z) = z^3$

8. Describe the Mayer-Vietoris sequence. Use it to calculate the homology groups of S^n and the Klein bottle K.