TOPOLOGY EXAM, August 2002.

1. Let $X = \{a, b, c\}$ and let $\mathcal{T}$ be a topology on X given by $\mathcal{T} = \{X, \phi, \{a,b\}, \{a,c\}, \{a\}\}$

Let $A = \{b, c\}$, $B = \{a, c\}$

Find (i) $\text{int}(A)$, $\text{int}(B)$

(ii) $\overline{A}$, $\overline{B}$

(iii) ∂A, ∂B

(iv) $\text{int}(A \cap B)$, $\overline{(A \cap B)}$

2. Define: X is locally compact

(b) Let W be the set $X \cup \infty$, where ∞ is a point not in X. Describe the topology on W that makes W the "one-point compactification" of X

(c) Prove that W is compact

(d) Show that the one point compactification $R^2 \cup \infty$ of R^2 is homeomorphic to the 2-dimensional sphere S^2.

3. Let X be a topological space and $I = [0,1]$ the unit interval. Let $T(X)$ be a space obtained from $X \times I$ by identifying the sets $X \times \{0\}$ and $X \times \{1\}$ to a single point x_0.

Show that the topological space $T(X)$ (the natural quotient topology on $T(X)$) is always connected.

4. Let X be (a): metric space
(b): Compact Hausdorff space.
Is X normal? If so prove this. If not construct a counterexample.

5. (a) Discuss the construction of $\pi_1(X, x_0)$
(b) Describe the homomorphism induced by $f : (X, x_0) \to (Y, y_0)$
(c) Explain why for the path connected space X, the fundamental group of X does not depend on the choice of a base point x_0

6. Write the answer (try to justified)

$\pi_1(R^3) \cong$

$\pi_1\left(R^3 \times S^1 \times \left(R^2 - \{pt\}\right)\right) \cong$

$\pi_1(RP^2) \cong$

$\pi_1(X) \cong$

where X is the Möbius band

7. Compute the homology groups ($\mathbb{Z}$ coefficients) for the torus $T^2 = S^1 \times S^1$ by:

ⓐ Mayer-Vietoris exact sequence

ⓑ using CW structure of T^2 and "cellular homology groups".

ⓒ Compute the homology groups for the Klein bottle. You can use M-V sequence or cellular homology.

8. Show that for each m, n R^m and R^n are homotopy equivalent but R^m not homeomorphic with R^n for $m \neq n$.

9. Let X be a space $S^1 \cup_f D^2$ i.e.

S^1 with attached 2-disk, where the attaching map $f : \partial D^2 = S^1 \longrightarrow S^1$ is given by
$$f(z) = z^4 .$$

Compute $H_*(X; Z)$