Topology Written Examination, Fall 2004

IN THIS EXAM ALL SPACES ARE ASSUMED TO BE HAUSDORFF

1. (a) Define what it means for X to be a compact space.

(b) Prove that in a compact space X each pair of disjoint closed sets A and B has disjoint open neighborhoods U and V.

(c) Let $f : X \to Y$ be a continuous bijection. Suppose X is compact. Show that f is a homeomorphism.

2. (a) Define what it means for X to be a connected space. Define what it means for X to be a path-connected space.

(b) Give an example of a connected space which is connected but not path connected. Sketch arguments on why your example is connected and why it is not path connected.

(c) Show that if U is an open connected subset of the plane, then U is path connected.

(d) Show that if X is connected, then the product space $X \times X$ is also connected.

3. (a) State the Seifert-van Kampen theorem.

(b) Use the Seifert-van Kampen theorem to compute the fundamental group of $\mathbb{RP}^2$ by decomposing it as the union of an Möbius band with a disk joined along the boundary of the Möbius band. Justify how the hypothesis of the Seifert-van Kampen theorem is satisfied.

(c) Compute the fundamental group of the projective plane by using the covering space of the 2-sphere over the projective plane. State clearly what relationship between covering spaces and fundamental groups you are applying to make this computation.

(d) Use your computation of the fundamental group to compute the first singular homology of the projective plane. State clearly the relationship between the fundamental group and the first singular homology that you are using, including a description of the map between these two groups used in establishing this relationship.

4. (a) Define a covering space (A, π, B) with covering map $\pi : A \to B$.

(b) Suppose (A, π, B) is a covering space and $f : X \to B$ is a continuous map with $f(x) = b, \pi(a) = b$. Give a necessary and sufficient condition in terms of the fundamental group that there exists a continuous lifting $\tilde{f} : X \to A$ with $\pi\tilde{f} = f$ with $\tilde{f}(x) = a$. Give a sketch of the proof that this condition is necessary and sufficient.

(c) Give a necessary and sufficient condition in terms of $\pi_1(A, a)$ and $\pi_1(B, b)$ that there is a lifting $\tilde{f}$ as above WITHOUT the condition that $\tilde{f}(x) = a$.

(d) Suppose X is simply connected. Show that any continuous map $f : X \to S^1$ is homotopic to a constant map.

5. (a) Define the cellular chain complex for a CW complex, including a description of how the boundary map is defined.

(b) Give the cellular chain complex for the projective plane $\mathbb{RP}^2$ and use it to compute its cellular homology.

(c) For each of the spaces $\mathbb{RP}^3$ and $\mathbb{RP}^2 \vee S^3$, compute their cellular homologies and show that they are isomorphic but use the homologies of their universal covering spaces to show that the two spaces are not homeomorphic.

6. (a) Describe how a short exact sequence of chain complexes leads to a long exact sequence in homology. Include a description of how the boundary map of the long exact sequence is defined.

(b) Apply the long exact sequence of the pair together with the excision property to deduce the homology of S^2 from the homology of S^1.

(c) Describe the Mayer-Vietoris exact sequence of a space X, including a careful statement of the hypotheses needed.

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