May 23, 2005
9AM - 1PM

TOPOLOGY QUALIFYING EXAM, SPRING 2005

There is a total of 10 problems.

1. Show that a closed subset of a compact topological space is compact.

2. Let X be Hausdorff and $A \subset X$ be a compact subset. Show A is closed.

3. Let X be Hausdorff and locally compact. Show that for any $p \in X$ and any open neighborhood U of p, there is an open neighborhood V of p such that $\overline{V} \subset U$ and $\overline{V}$ is compact.

4. Can you compute $\pi_1(S^1)$ using the Van Kampen theorem? If yes, give the details of your argument. If no, explain why.

5. Show that the unit sphere S^2 is a deformation retract of $\mathbb{R}^3 \setminus \{0\}$.

6. Show that the unit sphere $S^n \subset \mathbb{R}^{n+1}$ is canonically a smooth manifold.

7. (1) Compute the homology of $\mathbb{R}P^2$.

(2) Let $f : \mathbb{R}P^2 \to \mathbb{R}P^2$ be any continuous map. Compute the Lefschetz number $L(f)$.

(3) Use the Lefschetz fixed point theorem to show that f must have a fixed point.

8. Show that $\{(x, y, z) | x^2 + 2y^4 + 3z^4 = 1\}$ is a submanifold in $\mathbb{R}^3$.

9. Use the Mayer-Vietoris sequence and induction on the dimension n to compute the homology of S^n for all $n \geq 0$.

10. Let $X = S^2 \cup I$, where $S^2 = \{(x, y, z) | x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$ and $I = \{(0, 0, z) | z^2 \leq 1\} \subset \mathbb{R}^3$.

(1) Give X a CW-complex structure and compute its cellular homology.

(2) Consider $f : X \to X$ which is induced by $x \mapsto -x$, $y \mapsto y$ and $z \mapsto -z$. Compute the induced homomorphism of f on the homology of X.