TOPOLOGY EXAM

January 16, 2006   1:00pm – 5:00pm

All topological spaces are assumed to be Hausdorff.

1. Let $p : E \to B$ be a covering space. Let $p(e_0) = b_0$. Assume E is simply connected (i.e., path connected with a trivial fundamental group)
   (a) Define a bijection $\phi : \Pi_1(B, b_0) = p^{-1}(b_0)$
   (b) Use this bijection to prove $\Pi_1(P^2, b_0) \cong \mathbb{Z}_2$, where P^2 is the projective plane.
2. Define connected. Define path connected. Give an example of a space that is connected but not path connected. Prove that the product of two connected spaces is connected.
3. (a) Define compact, limit point compact, and sequentially compact.
   (b) What is the relation of compact to: limit point compact ?, to: sequentially compact ?
   (c) What if the space X in (b) is metric ?
   (d) Prove that a product of two compact spaces is compact.
4. (a) Define quotient map.
   (b) If $A \subset X$ is a retract of X onto A i.e., there is a map $r : X \to A$ with $r(a) = a$ for all $a \in A$, show that a retraction is a quotient map.
5. Define singular q–chains, q–cycles, and q–boundaries. Define $H_q(X)$. What is $H_n(S^n)$, $H_1(\mathbb{R}^3)$, and $H_1(\mathbb{R}P^3)$?
6. Show that $\mathbb{R}^n$ is homeomorphic with $\mathbb{R}^m \Leftrightarrow m = n$.
7. Compute the homology groups of the following spaces (the method is up to you):
   A. $X = T^2 = S^1 \times S^1$ (torus)
   B. $X = S^1 \underset{\phi}{\cup} D^2 \underset{\psi}{\cup} D^2$

   where ϕ is of degree 2, ψ of degree 3 (X_i is obtained from the circle by attaching two 2–cells).
8. Show that every continuous map $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point.
9. Discuss the Mayer-Vietoris sequence. Use it to compute homology groups of spheres (of all dimensions).