Topology Exam
August 27, 2007

1. Let $\mathbb{R}^n$ be n-dimensional Euclidean space with the usual topology. Let $X = \mathbb{R}^n \cup \{\infty\}$, where ∞ is a point not in $\mathbb{R}^n$.

a) Describe the topology on X that makes X the one-point compactification of $\mathbb{R}^n$.

b) Show that X is homeomorphic to S^n, the n-dimensional sphere.

2. A space X is locally path connected if for every point x of X and every neighborhood U of x, there is a path-connected neighborhood V of x contained in U.

a) Give an example of a space that is connected, but not path connected.

b) Show that a space X that is connected and locally path connected is path connected.

3. Sketch a proof that the fundamental group of the circle is infinite cyclic.

4. Sketch a proof that $\pi_1(S^2)$ is trivial. Then prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$.

5. Let Y be the “figure 8”. Give 3 examples of covering spaces of Y. Use one of them to prove that $\pi_1(Y, y_0)$ is not abelian.

6. Consider the space X which is the union of the unit sphere S^2 in $\mathbb{R}^3$ and the line segment between the north and south poles.

(a) Give it the structure of a CW complex. (Hint:The north and south poles should be the 0 -cells.)

(b) Compute its homology, using this structure.

(c) Show that X is homotopy equivalent to the one point union $S^1 \vee S^2$ of a 2-sphere and a circle. Use this result to redo (b).

7. For each of the following spaces, give π_1 and H_* (Z-coefficients for H_*). Indicate how you are getting your answer - a full proof is not required.

(a) S^1

(b) S^n, $n > 1$

(c) $\mathbb{R}P^2$ (real projective plane)

(d) $\mathbb{C}P^2$

(e) $\mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$

(f) $\mathbb{R}^2$ - $\{(0,0) \cup (0,1)\}$

8. Use the Mayer-Victoris sequence to compute the homology groups of the space obtained from a torus by attaching a Mobius band via a homeomophism from the boundary circle of the Mobius band to the circle S^1 x $\{x_0\}$ in the torus.