

GEOMETRY OF DEL PEZZO SURFACES

The goal of this course is to acquaint students with the techniques of algebraic geometry in a fairly concrete setting. The del Pezzo surfaces are constructed from a basic surgery operation, known as a blow-up. To blow-up a plane at a marked point, remove the marked point and replace it with a line. The key will be to glue this line into the plane in a way that we still get a manifold; the main idea behind the construction is that the points on this new line that we add correspond to the different directions from the marked point in the plane. The del Pezzo surfaces are obtained by applying this surgery operation to a configuration of up to 8 points in the plane that are in a sufficiently “general” position.

After learning about the blow-up and using it to construct del Pezzo surfaces, we will turn to studying their geometry. What does this mean? Well, here are some basic geometric properties that we will study:

- (1) How can we embed these surfaces into a projective space? It turns out we can then think of the del Pezzo surfaces as subsets of projective space given as the zero locus of polynomials. And so we may ask: what kind of polynomial equations define del Pezzo surfaces?
- (2) Another way to study the geometry of a surface is to look at the curves that lie on the surface. Inspired by topology, we develop an intersection theory for curves on a surface. Since our del Pezzo surfaces can be embedded into projective space, we can study whether any lines in projective space are completely contained in our surface. Amazingly, the answer is yes, and there are finitely many of them! How many? How are they configured?
- (3) Related to the study of curves on a surface is the study of line bundles on the surface. We will explore why these two seemingly different topics are actually closely related; this observation will help us to refine our study of the first two questions.
- (4) One of the major interests in the study of del Pezzo surfaces is the existence of rich symmetry in their geometry. For example, mathematicians have long been fascinated by the symmetries of the configuration of lines on del Pezzo surfaces that respect the intersection properties of the lines.

By the end of the course, you will be able to explain why a cubic surface in complex projective three-space (a particularly concrete example of a del Pezzo surface) has a configuration of exactly 27 lines with a symmetry group of order 51,840.

Time permitting, the course will include a brief discussion of some arithmetic questions about del Pezzo surfaces and a conjectural solution to one of these problems which makes extensive use of their geometry.

Reading

Reading for this course will come from a number of sources. For background on algebraic geometry, a readable reference is Igor Shafarevich's *Basic Algebraic Geometry I*. A more gentle introduction can be found in Karen Smith's *An Invitation to Algebraic Geometry* and Miles Reid's *Undergraduate Algebraic Geometry*. The latter book finishes with a nice discussion of cubic surfaces and their lines. References on algebraic surfaces include Arnaud Beauville's *Complex Algebraic Surfaces*, a comprehensive survey of their classification that requires a fair amount of background to read, and Reid's *Chapters on Algebraic Surfaces*, which requires less background knowledge and includes a discussion of cubic surfaces.

The primary reading source for the course will be Yu. Manin's *Cubic Forms*. Also of interest, for those who read French, are M. Demazure's articles on del Pezzo surfaces in the *Seminaire sur les Singularites des Surfaces*. I plan to supplement some of this reading with personal notes that include any necessary background material.

Hopefully, we may also get to discuss some recent results in the literature dealing with del Pezzo surfaces. The following articles, all available at <http://www.arxiv.org>, are of particular interest:

- Victor Batyrev and Oleg Popov, "The Cox ring of a del Pezzo surface"
- Brendan Hassett and Yuri Tschinkel, "Universal torsors and Cox rings"
- Tim Browning, "An overview of Manin's conjecture for del Pezzo surfaces"