Comments. This set of questions should help you prepare for the final exam. I expect the final to be similar in length and style of questions. In addition, make sure that you're comfortable with the problems and solutions from the first two midterms. As usual, calculators, *etc.*, may not be used for the exam and you should show all of your work.

Question 1. Suppose the variable x represents students, y represents courses, and:

- $M(y)$ means y is a math course
- $B(x)$ means x is a full-time student
- $F(x)$ means x is a freshman
- $T(x, y)$ means x is taking y

Write the following as English sentences (without variables or other symbols):

- $\forall x \exists y T(x, y)$
- $\exists x \forall y T(x, y)$
- $\forall x \exists y [(B(x) \wedge F(x)) \rightarrow (M(x) \wedge T(x, y))]$

Question 2. Let $A = \{1, 2\}$ and $B = \{1, \{2\}\}$.

- Compute $A \cup B$
- Compute $A \cap B$
- Compute $A \times B$
- Compute $A - B$
- Compute $A \Delta B$
- Compute $\mathcal{P}(B)$ (the power set of B)

Question 3. Let m be a positive integer. For integers a, b, c, and d, prove that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $a + c \equiv b + d \pmod{m}$. (Try a direct proof.)

Question 4. Prove that $f(n) = 5 \cdot 1^4 + 5 \cdot 2^4 + 5 \cdot 3^4 + \cdots + 5 \cdot n^4 + \frac{4n^6+2n^3}{n^3+1} = O(n^5)$ as $n \to \infty$. (Exhibit a k and C as required by the definition.)

Question 5. Let $\{a_n\}$ be the sequence defined by

$$a_n = a_{n-1}^2 + 1$$

and $a_0 = 1$. What is the value of a_3?

Question 6. Let $a_1 = 2$, $a_2 = 9$, and $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. Prove that $a_n \leq 3^n$ for all positive integers n. [Try induction.]

Question 7. Consider license plates with strings of eight numbers (0-9) and letters (A-Z). How many license plates have no letters? How many have at most one letter? How many have at most two letters?

Question 8. Let A be a set with 5 elements and B a set with 4 elements.

- How many functions are there from A to B?
- How many functions from B to A are injective (one-to-one)?
- How many functions from A to B are surjective (onto)?

Question 9.

- Find a recurrence relation for the number of ways to climb n stair s if stairs can be climbed two or three at a time.
- What are the initial conditions?
- How many ways are there to climb eight stairs?

Question 10. What is the solution to the recurrence relation $a_n = 8a_{n-1} + 9a_{n-2}$ if $a_0 = 3$ and $a_1 = 7$?

Question 11. How many permutations of the alphabet (A–Z) contain at least one of the words CART, SHOW, and LIKE?

Question 12. Show that the relation R, consisting of all pairs (x, y) where x and y are bitstrings of length ≥ 3 and whose bits (with the possible exception of the first three bits) agree is an equivalence relation on the set of all bitstrings of length ≥ 3.

Question 13. Let R be a relation on a set with four elements, and its matrix given by

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

- Is R reflexive?
- Is R symmetric?
- Is R antisymmetric?
- Is R transitive?

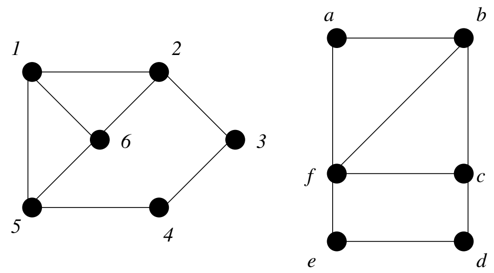
Justify each of your answers.

Question 14. Suppose $A = \{2, 3, 6, 9, 10, 12, 14, 18, 20\}$ and R is a partial order on A defined by xRy iff x is a divisor of y.

- Draw the Hasse diagram for R.
- Find all maximal elements.
- Find all minimal elements.
- Find the least upper bound of $\{2, 9\}$.
- Find the least upper bound of $\{3, 10\}$.
- Find the greatest lower bound of $\{14, 10\}$.
- Give an example of an incomparable pair of elements from A.

Question 15. Consider the following two graphs.

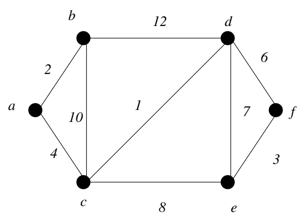
- Are they isomorphic? (Justify your answer.)
- Write the adjacency matrix of the graph on the left.



Question 16.

- What is the difference between Euler and Hamilton paths?
- Draw a copy of K_5. Give an example of an Euler circuit in it and an example of a Hamilton path between two of its vertices.

Question 17. Find a minimum spanning tree in the following weighted graph.



Question 18. How many edges must a connected graph on n vertices have? Justify your answer.