Pattern Formation by Bacteria-Driven Flow

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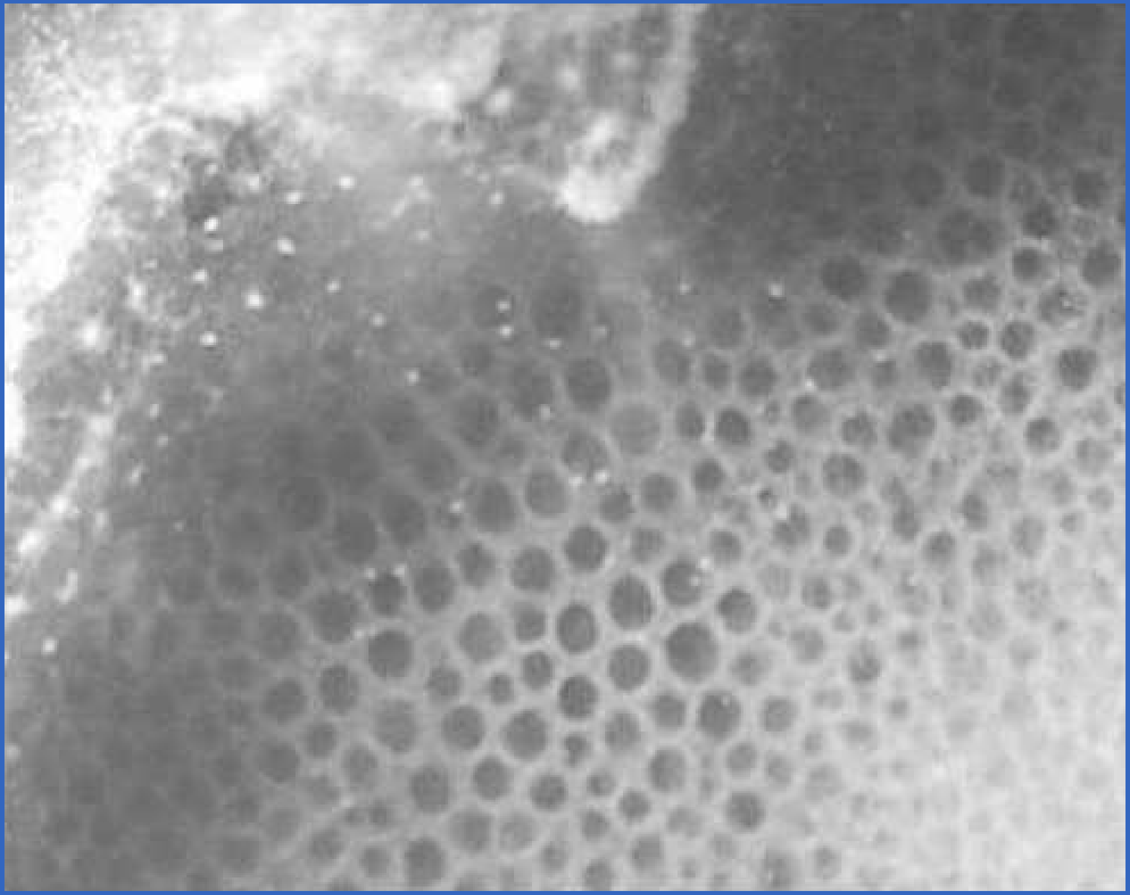
- Dr. Charles Wolgemuth - University of Connecticut, Health Center
- Dr. Roland Thar - University of Copenhagen, Marine Biological Laboratory

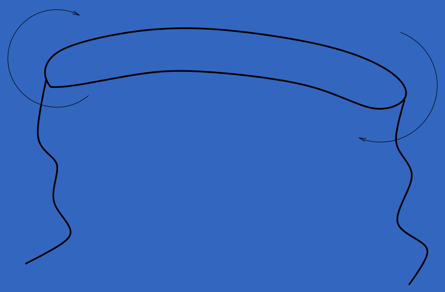
Outline

- Physical/Biological Background
- Mathematical Model
- One-Dimensional Results
- Two-Dimensional Results
- Conclusions

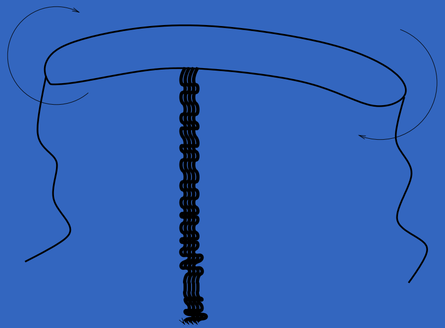
Beginning of the Story



Players



- Vibroid bacteria
- Chemotaxis
  - Sulfide - marine sulfide sediment
  - Oxygen - limiting nutrient
- Mucus - Produced by and anchors bacteria



Storyline

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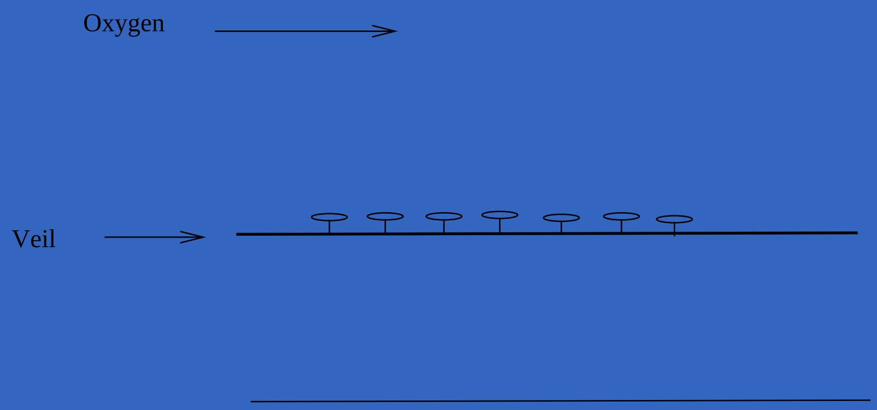

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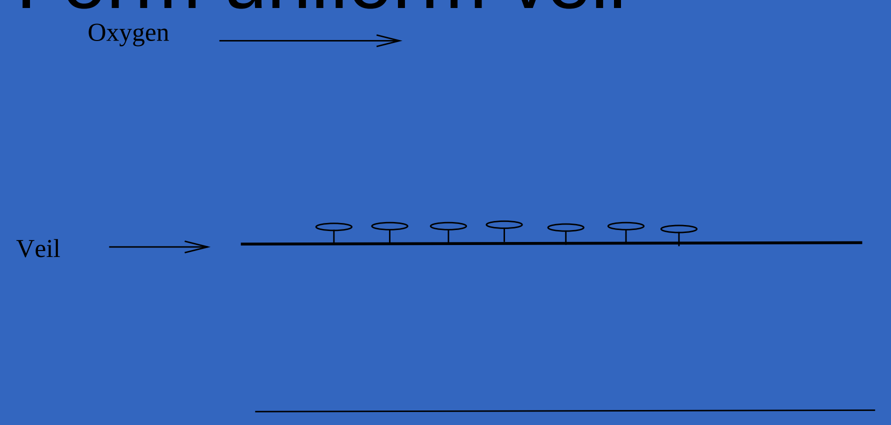

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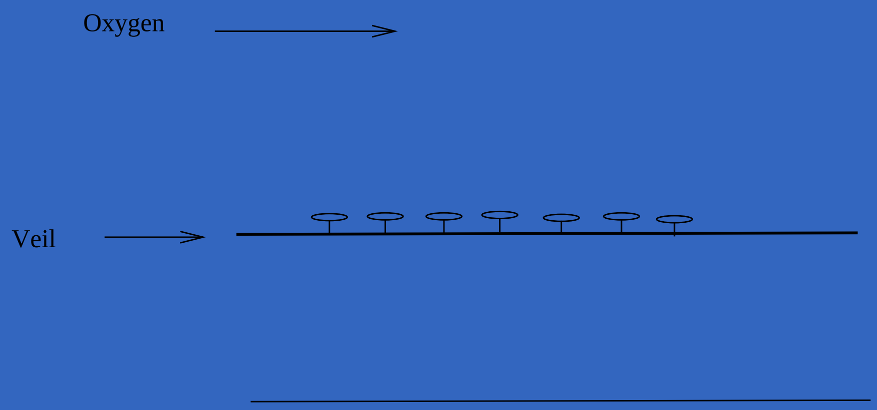

- Consume Oxygen
- Rotate Flagella, pump oxygenated fluid

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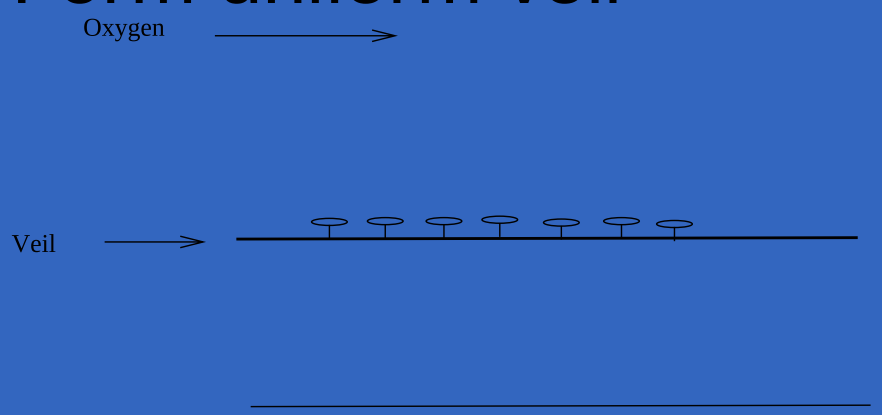
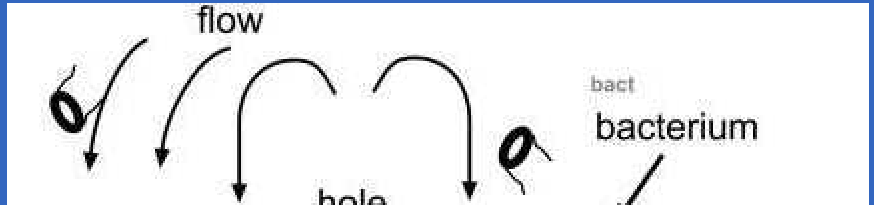


- Consume Oxygen
- Rotate Flagella, pump oxygenated fluid
- Detach/re-attach



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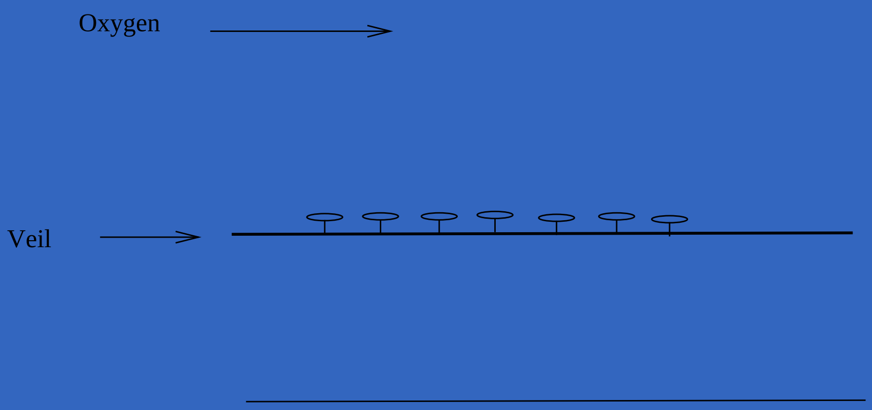

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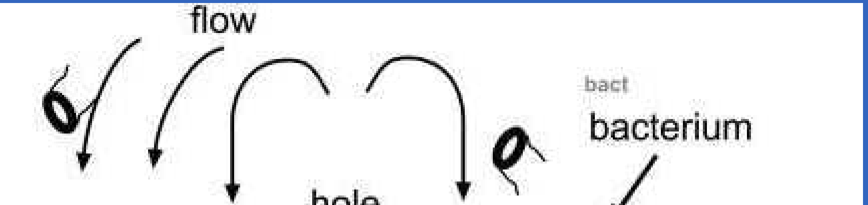


Simplifying Assumptions 1D veil in 2D fluid

- Bacteria - Free: n_f and Bound: n_b
- Veil height constant: h
- Bound cells break, time-scale: τ_b
- Free bacteria swimming, diffusive: D_f
- Detached cells swim average height, δ, above veil for τ_f

Model Equations

- Cells

$$\frac{\partial n_b}{\partial t} = -\frac{1}{\tau_b} n_b + \frac{1}{\tau_f} n_f$$

$$\frac{\partial n_f}{\partial t} = D_f \frac{\partial^2 n_f}{\partial x^2} - \frac{\partial (n_f \mathbf{v}_{x,\delta})}{\partial x} + \frac{1}{\tau_b} n_b - \frac{1}{\tau_f} n_f$$

- Fluid Dynamics

$$\eta \Delta \mathbf{v} - \nabla p = 0$$

$$\nabla \cdot \mathbf{v} = 0$$

Fluid Velocity

- Bound bacteria force fluid $\mathbf{K} = -\alpha n_b \hat{\mathbf{y}}$
- Fluid Flow, i^{th} direction: Superposition of singular solutions

$$\mathbf{v}_i = \frac{1}{4\pi\eta} \int dx_0 \mathbf{G}_{ij}(\mathbf{x}, bx_0)\mathbf{K}_j$$

Fluid Velocity cont'd

- Greens Function:

$$\mathbf{G}_{ij}(\mathbf{x},\mathbf{x}_0) = \mathbf{S}_{ij}(\mathbf{X}) - \mathbf{S}_{ij}(\mathbf{X}^I) + 2h^2\mathbf{G}_{ij}^{D}(\mathbf{X}^I) - 2h\mathbf{G}_{ij}^{SD}(\mathbf{X}^I)$$

- $\mathbf{X} = \mathbf{x} - \mathbf{x}_0$, $\mathbf{X}^I = \mathbf{x} - \mathbf{x}_0^I$ and $\mathbf{x}_0^I = (x_0, -h)$
- Velocity δ units above the veil:

$$\mathbf{v}_{x,\delta} = -\frac{\alpha}{4\pi\eta}\int_{-\infty}^{\infty} dx_0 n_b(x_0) G_{xy}$$

Linear Stability

- Expand homogeneous solutions

$$n_{b,f} = n_{b,f}^{(0)} + \epsilon n_{b,f}^{(1)} e^{\gamma t} \cos(qx)$$

- Substitute into non-dimensional equations:

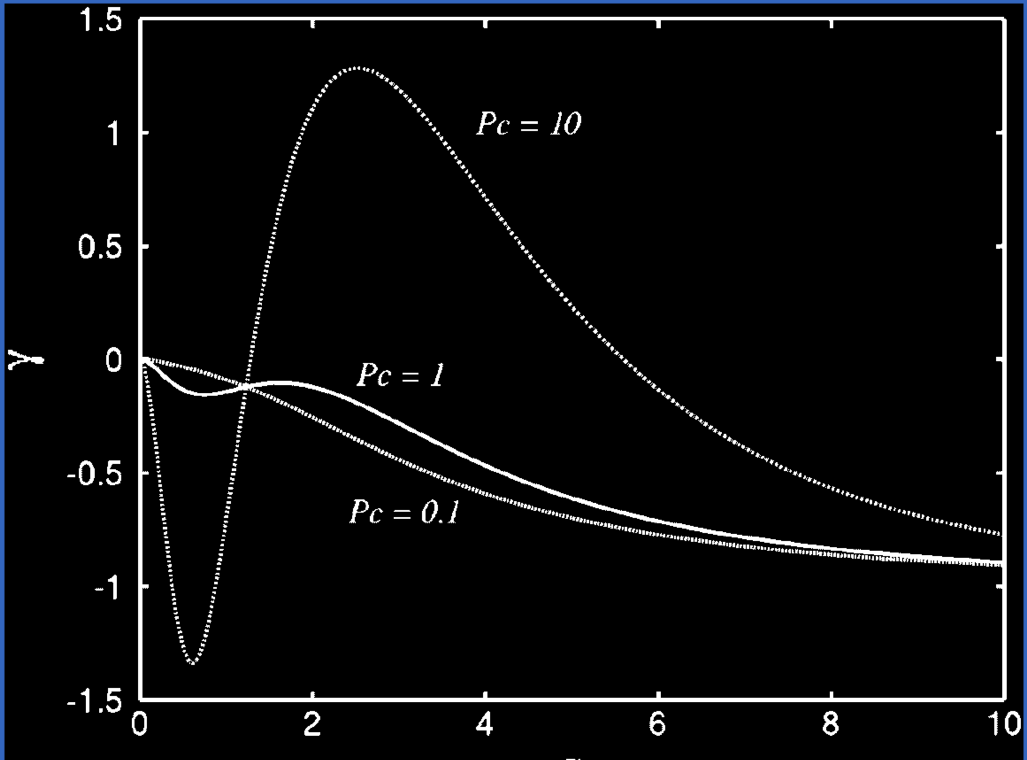
$$\begin{aligned} (\gamma + x_-) n_b^{(1)} - x_+ n_f^{(1)} &= 0 \\ -(x_- + P_c \theta) n_b^{(1)} + (\gamma + q^2 + x_+) n_f^{(1)} &= 0 \end{aligned}$$

Non-dimensional terms

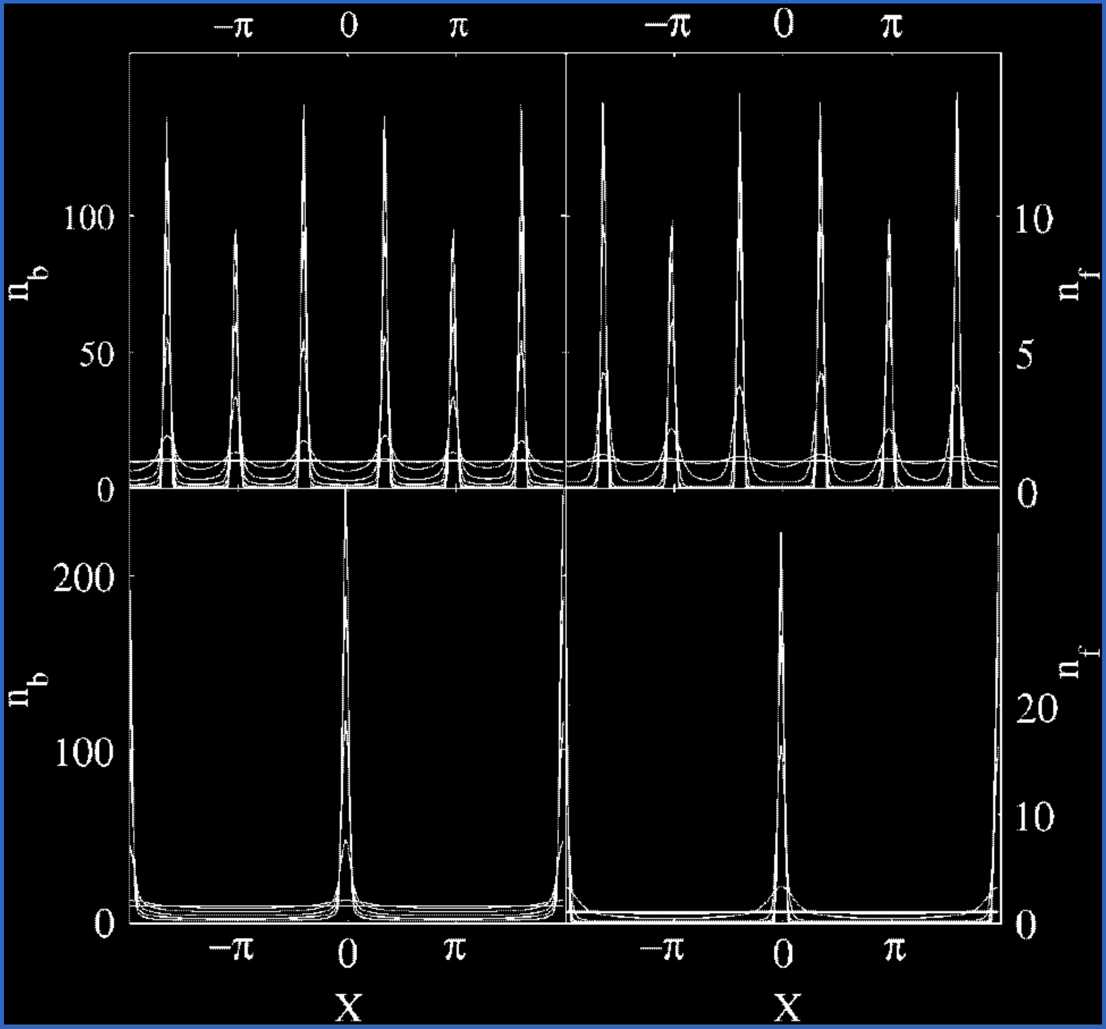
- k_- and k_+: Non-dimensional residence times
- P_c: Peclet number (advection vs. diffusion)
- θ: first-order advection term (*Analytic expression*)

Dispersion curve

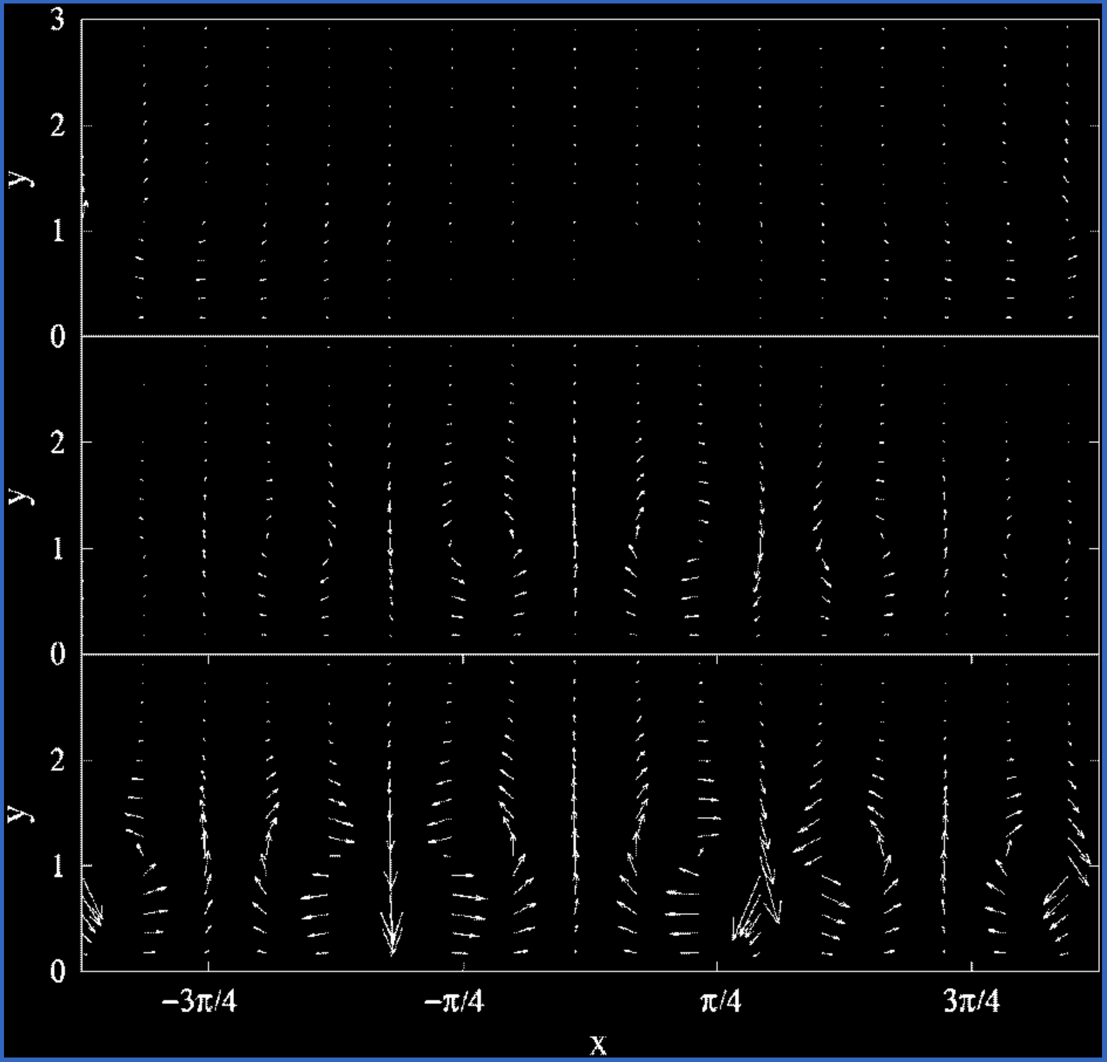
$$\gamma = \frac{1}{2}\left(\left(q^2 + \frac{1}{\tau_f} + \frac{1}{\tau_b}\right)^2 + 4\left(\frac{1}{\tau_f}\theta - \frac{1}{\tau_b}q^2\right)\right)^{(1/2)} - \frac{1}{2}\left(q^2 + \frac{1}{\tau_f} + \frac{1}{\tau_b}\right)$$



Bacterial Aggregation (full eqns)



Flow development



2D veil immersed in 3D fluid

- Cells

$$
\begin{aligned}
\frac{\partial n_b}{\partial t} &= -\frac{1}{\tau_b} - n_b + \frac{1}{\tau_f} n_f \\
\frac{\partial n_f}{\partial t} &= D_f \Delta n_f - \left(\frac{\partial n_f \mathbf{v}_{x,\delta}}{\partial x} + \frac{\partial n_f \mathbf{v}_{z,\delta}}{\partial z}\right) \\
&\quad + \frac{1}{\tau_b} n_b - \frac{1}{\tau_f} n_f
\end{aligned}
$$

- Velocities:

$$
\mathbf{v}_i = \frac{1}{8\pi\eta} \int d\mathbf{x}_0 \mathbf{G}_{ij}(\mathbf{x}, \mathbf{x}_0) \mathbf{K}_j
$$

Greens Function

$$\mathbf{G}_{ij} = \mathbf{S}_{ij}(\mathbf{X}) - \mathbf{S}_{ij}(\mathbf{X}^I) + 2h^2\mathbf{G}_{ij}^{D}(\mathbf{X}^I) - 2h\mathbf{G}_{ij}^{SD}(\mathbf{X}^I)$$

where $\mathbf{X} = \mathbf{x} - \mathbf{x}_0$, $\mathbf{X}^I = \mathbf{x} - \mathbf{x}_0^I$, and $\mathbf{X}_0^I = (x_0, -h, z_0)$

Stokeslets, Doublets etc.

$$\mathbf{S}_{ij}(\mathbf{x}) = \frac{\delta_{ij}}{|\mathbf{x}|} + \frac{x_i x_j}{|\mathbf{x}|^3}$$

$$\mathbf{G}D_{ij}(\mathbf{x}) = \pm\left(\frac{\delta_{ij}}{|\mathbf{x}|^3} - 3\frac{x_i x_j}{|\mathbf{x}|^5}\right)$$

$$\mathbf{G}_{ij}^{SD}(\mathbf{x}) = x_2 \mathbf{G}_{ij}^{D}(\mathbf{x}) \pm \frac{\delta_{j2} x_i - \delta_{i2} x_j}{|\mathbf{x}|^3}$$

where the plus sign is taken for $j = 1, 2$ and the minus sign for $j = 3$.

Linearization

- Expand

$$n_{f,b} = n_{f,b}^{(0)} + \epsilon n_{f,b}^{(1)} e^{i(\mathbf{q}\mathbf{x})+\gamma t}$$

- $\mathbf{q} = (q_1, 0, q_2)$ - independent of y
- Linear system (non-dimensional):

$$(\gamma n_b^1 + k_- n_b^1 - k_+ n_f^1) e^{i(q_1 x + q_2 z)+\gamma t} = 0$$

$$\left((\gamma + q_1^2 + q_2^2 + k_+) n_f^1 - k_- n_b^1\right) e^{i(q_1 x + q_2 z)+\gamma t} - P_c n_f^0 \left(\frac{\partial}{\partial x} v_{x,\delta}^{(1)} + \frac{\partial}{\partial z} v_{z,\delta}^{(1)} \right) = 0$$

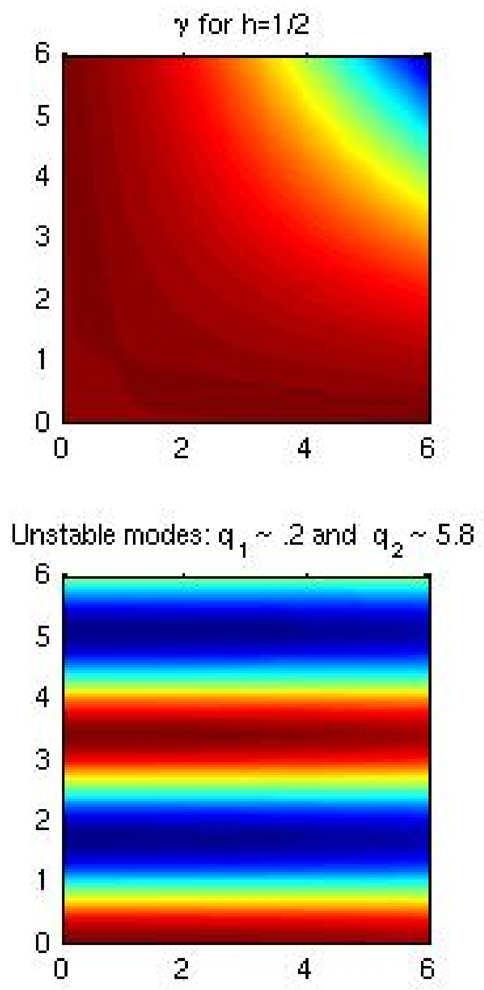
Evaluation of Velocities

- In 1D, closed form for velocity
- Unable to do so in 2D
- Expand Greens functions in power series
- dispersion relation:

$$\begin{aligned}(\gamma + k_-)(\gamma + q_1^2 + q_2^2 + k_+)\\ + k_+\left(k_- - \frac{P_c n_f^{(0)}}{16\pi^2} I(q_1\theta_1 + q_2\theta_2)\right) &= 0\end{aligned}$$

- θ_1 and θ_2 denote contributions from the x and z velocities

Dispersion Relation



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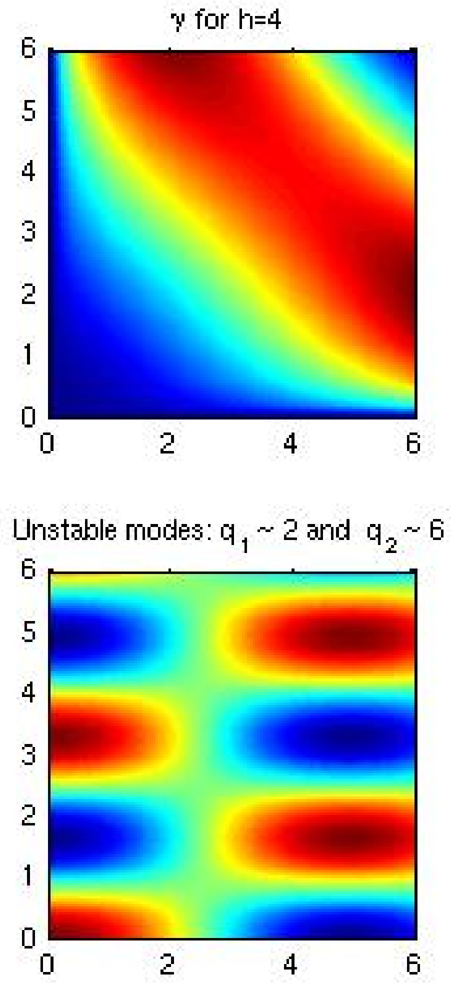

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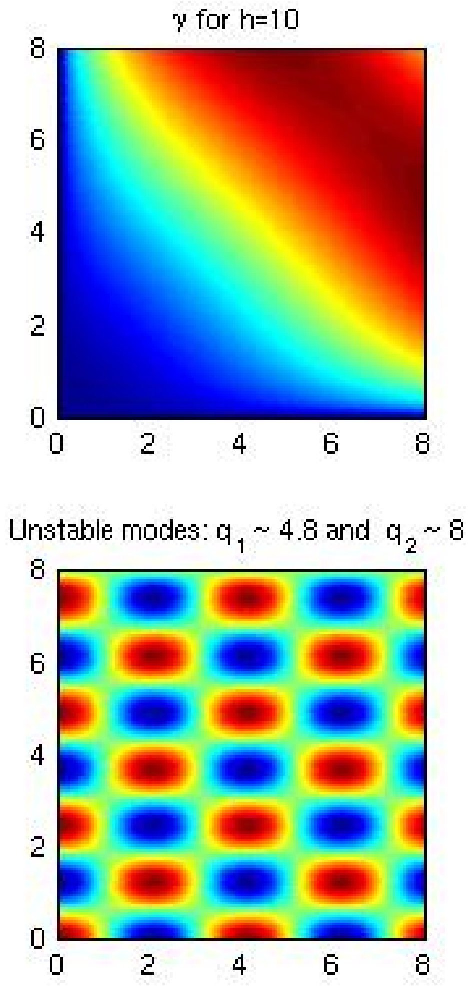

Conclusions

- Model captures scales of patterns (1 and 2D analysis)
- Rolls vs. Cells controlled by height of veil
- Explicit Chemotaxis
- Assumed fixed height
- Assumed holes from degradation - fluid interactions?
- 2D nonlinear analysis