Image system for Stokes-flow singularity between two parallel planar walls

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Using a recently developed image representation for Stokes flow in a half-space bounded by a planar wall [Cichocki and Jones, Physica A **258**, 273 (1998)], the image system is constructed for the flow field produced by a force multipole in the space bounded by two parallel walls. The image singularities are expressed in terms of products of double-reflection matrices, and the expansion is simplified using symmetries of the double-reflection operation. Our analysis yields recurrence relations for the strengths of the image multipoles. The relations are solved explicitly, and a complete image system is obtained for an arbitrary source-force multipole. Applications of our image representation for evaluating the hydrodynamic friction and mobility matrices of particles interacting with two parallel planar walls are indicated. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1508812]

I. INTRODUCTION

Dynamics of particles suspended in a viscous fluid that occupies the space bounded by a planar interface[1–8] or two planar walls[9–15] has recently attracted much attention. Hydrodynamic interactions of particles in such geometries are affected by the flow field reflected from the walls. An important tool for investigating particle motion in wall-bounded systems is the image representation of the reflected flow.[16]

The Stokes flow reflected from a single no-slip wall was originally discussed by Lorentz,[17] and the image of a Stokeslet was derived by Blake.[18] Recently, a complete image representation for a force multipole of an arbitrary order was obtained by Cichocki and Jones.[4] This image representation was used to determine single-particle[4] and many-particle[8] friction and mobility coefficients for the motion of spherical particles near a rigid wall.

In the present article we derive the image representation for a flow field generated by a force multipole between two parallel rigid walls. In an earlier study,[19] the image solution was considered only for a special case of Stokeslet, but a complete multiple-reflection singularity representation was not found. The difficulty of the analysis stems from the form of the one-wall solution: unlike the reflection of an electrostatic charge multipole from a conducting plane, the image of a force multipole of an order l involves a combination of multipoles of orders $l' \neq l$. For this reason, the multiple-reflection series for Stokes flow in a space bounded by two parallel walls involves complicated combinations of image multipoles of different orders and strengths, placed at the positions corresponding to multiple mirror reflections of the source in the bounding planes.

A key ingredient of our analysis is the double-reflection identity that results from a symmetry between the source and the image flows in a system with a single reflection plane. Both of these flows satisfy Stokes equations, and their sum vanishes on the plane; thus, the reflection operation applied twice, first to the source and then to the image flow, yields the original source flow. The double-reflection identities are used to simplify the multiple-reflection sequences in a two-wall system and to derive simple recurrence relations for the strength of the image force multipoles. The recurrence relations are solved explicitly.

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This article is organized as follows. In Sec. II, basic notation is introduced, and image singularities are represented in terms of a complete set of Stokes-flow fields. The image representation for a force multipole in a system with a single wall is discussed in Sec. III, where the double-reflection identities are derived. The identities are used in Sec. IV to obtain the image solution for a force multipole between two walls. The results are discussed in Sec. V.

II. FORMULATION

We consider Stokes flow produced by a multipolar force distribution $\mathbf{F}$ located at $\mathbf{r}=0$ in a region bounded by two infinite parallel planar walls (a) and (b) located at

$$z=-h^{(a)}, \quad z=h^{(b)}. \tag{2.1}$$

At the walls, the flow field $\mathbf{u}$ satisfies no-slip boundary conditions.

The incident flow $\mathbf{v}$, produced by the force distribution $\mathbf{F}$ in infinite space, can be expanded in a complete set of basis solutions of Stokes equations. We choose here the representation introduced by Cichocki *et al.*,[20]

$$\mathbf{v}(\mathbf{r})=\sum_{lm\sigma} f_{lm\sigma}\mathbf{v}^{-}_{lm\sigma}(\mathbf{r}), \tag{2.2}$$

where $f_{lm\sigma}$ are the multipole moments of the distribution $\mathbf{F}$, and the basis velocity fields are of the form

$$\mathbf{v}^{-}_{lm\sigma}(\mathbf{r})=\mathbf{V}_{lm\sigma}(\theta,\phi)r^{-(l+\sigma)}. \tag{2.3}$$

Here (r,θ,ϕ) represent vector $\mathbf{r}$ in spherical coordinates, and the indices assume values $l=1,2,\ldots$; $m=-l,\ldots,l$; and $\sigma=0,1,2$. The functions $\mathbf{V}_{lm\sigma}(\theta,\phi)$ are combinations of vector spherical harmonics with angular and azimuthal quantum numbers l and m. This property and the r-dependence in Eq. (2.3) define the functions $\mathbf{v}^{-}_{lm\sigma}(\mathbf{r})$ up to a normalization constant; we use here the same normalization as in Ref. 20. Explicit expressions for the functions $\mathbf{V}_{lm\sigma}(\theta,\phi)$ are listed in Appendix A.

In the presence of the walls, the reflected flow field

$$\mathbf{v}_{\mathrm{s}}=\mathbf{u}-\mathbf{v} \tag{2.4}$$

can be represented in terms of image singularities. In a system with the source singularity at $\mathbf{r}_{\mathrm{s}}=z_{\mathrm{s}}\mathbf{e}_z$ and a single wall at $z=z_{\mathrm{w}}$, the image singularity is at $\mathbf{r}_{\mathrm{i}}=(2z_{\mathrm{w}}-z_{\mathrm{s}})\mathbf{e}_z$, and

$$\mathbf{v}_{\mathrm{s}}(\mathbf{r})=\sum_{lm\sigma} f^{*}_{lm\sigma}\mathbf{v}^{-}_{lm\sigma}(\mathbf{r}-\mathbf{r}_{\mathrm{i}}), \tag{2.5}$$

where $f^{*}_{lm\sigma}$ are the multipole moments of the image, and $\mathbf{e}_z$ is the unit vector in the z direction.

For two walls, the image representation can be obtained from (2.5) in a form of a multiple-reflection sequence. Accordingly,

$$\mathbf{v}_{\mathrm{s}}(\mathbf{r})=\sum_{\alpha=a,b}\sum_{i=1}^{\infty}\mathbf{v}_{\mathrm{s}}^{(i\alpha)}(\mathbf{r}), \tag{2.6}$$

where

$$\mathbf{v}_{\mathrm{s}}^{(i\alpha)}(\mathbf{r})=\sum_{lm\sigma} f^{(i\alpha)}_{lm\sigma}\mathbf{v}^{-}_{lm\sigma}(\mathbf{r}-\mathbf{r}^{(i\alpha)}). \tag{2.7}$$

For the geometry (2.1), the positions of the image singularities are $\mathbf{r}^{(i\alpha)}=H^{(i\alpha)}\mathbf{e}_z$, with

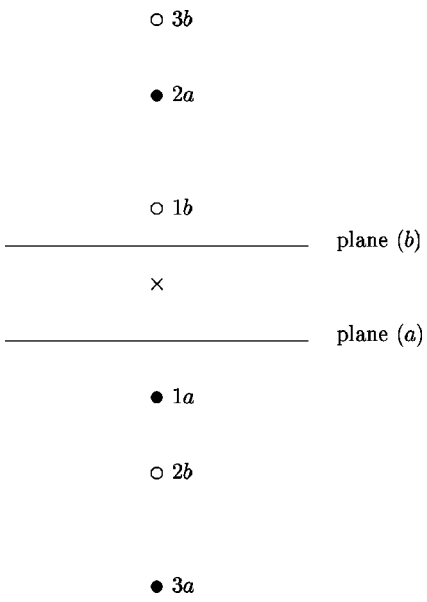


FIG. 1. Positions $\mathbf{r}^{(ia)}$ and $\mathbf{r}^{(ib)}$ $(i=1,2,\ldots)$ of image singularities in two families (a) and (b).

$$\begin{aligned} H^{(2r\,a)}&=2rh, \quad H^{(2r+1\,a)}=-2(h^{(a)}+rh),\\ H^{(2r\,b)}&=-2rh, \quad H^{(2r+1\,b)}=2(h^{(b)}+rh), \end{aligned} \tag{2.8}$$

where $h=h^{(a)}+h^{(b)}$ is the distance between walls, $H^{(0a)}=H^{(0b)}=0$ describes the position of the source singularity, and the index $r=0,1,2...$ characterizes the double-reflection order. The family of images at $\mathbf{r}^{(i\alpha)}$, $i=1,2,\ldots$, corresponds to the multiple-reflection sequence with the initial reflection on the plane (α), as illustrated in Fig. 1. The explicit expressions for the multipole moments $f^{(i\alpha)}_{lm\sigma}$ are derived in the following sections.

III. SINGLE-REFLECTION MATRIX

A. Multipoles of image force

By linearity of Stokes equations, the multipole moments of the source and image force distributions are linearly related,

$$f^{*}_{lm\sigma}=\sum_{l'm'\sigma'} R(H,lm\sigma|l'm'\sigma')f_{l'm'\sigma'}, \tag{3.1}$$

where R is the reflection matrix, and $H=z_{\mathrm{s}}-z_{\mathrm{w}}$ describes the relative position of the source singularity and the wall. The explicit form of the reflection matrix has recently been derived by Cichocki and Jones,[4] and we follow here their notation.

The successive images in the families $\alpha=a,b$ are obtained by a sequence of reflections

$$f^{(i\alpha)}_{lm\sigma}=\sum_{l'm'\sigma'} R(H^{(i-1\alpha)}_{\mathrm{w}},lm\sigma|l'm'\sigma')f^{(i-1\alpha)}_{l'm'\sigma'}, \quad i=1,2,\ldots, \tag{3.2}$$

where

$$H^{(i\alpha)}_{\mathrm{w}}=\tfrac{1}{2}(H^{(i\alpha)}-H^{(i+1\alpha)}). \tag{3.3}$$

The reflection is on the plane α for $i=2r+1$ and at the plane $\beta\neq\alpha$ for $i=2r$.

B. Structure of reflection matrix

Due to the rotational symmetry with respect to the axis z, the matrix R is diagonal in the angular quantum number m,

$$R(H,lm\sigma|l'm'\sigma')=R(H,lm\sigma|l'm\sigma')\,\delta_{mm'}. \tag{3.4}$$

The dependence of $R(H,lm\sigma|l'm'\sigma')$ on the relative coordinate H and the azimuthal quantum numbers l, l' can be derived from the Lorentz's reflection formula[16,17] for the reflected field,

$$\mathbf{v}_s=\hat{\mathrm{P}}(\hat{\mathbf{R}}_0+H\hat{\mathbf{R}}_1+H^2\hat{\mathbf{R}}_2)\cdot\mathbf{v}, \tag{3.5}$$

where

$$\hat{\mathbf{R}}_0=-\mathbf{I}_z-2\bar{z}\boldsymbol{\nabla}\mathbf{e}_z+\bar{z}^2\nabla^2\mathbf{I}, \tag{3.6}$$

$$\hat{\mathbf{R}}_1=-2\boldsymbol{\nabla}\mathbf{e}_z+2\bar{z}\nabla^2\mathbf{I}, \tag{3.7}$$

$$\hat{\mathbf{R}}_2=\nabla^2\mathbf{I}. \tag{3.8}$$

In the above relations, $\bar{z}=z-z_s$ is the coordinate relative to the source position, $\mathbf{I}$ is the identity tensor, $\mathbf{I}_z=\mathbf{I}-2\mathbf{e}_z\mathbf{e}_z$, and $\hat{\mathrm{P}}$ is the reflection operator with respect to the plane $z=z_w$,

$$[\hat{\mathrm{P}}\mathbf{w}](x,y,z)=\mathbf{I}_z\cdot\mathbf{w}(x,y,2z_w-z). \tag{3.9}$$

Since the operators (3.6)–(3.8) can be decomposed into a linear combination of tensor operators with azimuthal quantum numbers $l''\leqslant 2$, the matrix elements (3.4) vanish for $|l-l'|>2$, because of the triangle property of Clebsch–Gordan coefficients.[21] Moreover, operators $\hat{\mathbf{R}}_i$ are homogeneous of the order $-i$ in $\mathbf{r}$, according to Eqs. (3.6)–(3.8). It thus follows from relations (2.3) and (3.5) that

$$\begin{aligned}R(H,n-\sigma\ m\ \sigma|n'-\sigma'\ m\ \sigma')=&R_{n\,n}(\sigma\sigma',m)\,\delta_{n'n}+HR_{n\,n-1}(\sigma\sigma',m)\,\delta_{n'n-1}\\&+H^2R_{n\,n-2}(\sigma\sigma',m)\,\delta_{n'n-2}.\end{aligned} \tag{3.10}$$

The explicit expressions for the matrix elements $R_{n\,n'}(\sigma\sigma',m)$ are listed in Appendix B.

C. Reduced matrix notation

Relations (3.4) and (3.10) allow us to introduce a compact matrix notation. Accordingly, we define a vector space $\mathcal{S}$ of infinite column vectors f with components f_n $(n=1,2,\ldots)$, where each component itself consists of three components $\mathsf{f}_n(\sigma)$ $(\sigma=0,1,2)$. We also define matrices A acting in $\mathcal{S}$: each element $\mathsf{A}_{n\,n'}$ of such a matrix is itself a three-dimensional matrix with the elements $\mathsf{A}_{n\,n'}(\sigma\sigma')$. In the space $\mathcal{S}$, the products Af and AB are defined in a natural way,

$$(\mathsf{Af})_n=\sum_{n'=1}^{\infty}\mathsf{A}_{n\,n'}\mathsf{f}_{n'},\quad (\mathsf{AB})_{n\,n''}=\sum_{n'=1}^{\infty}\mathsf{A}_{n\,n'}\mathsf{B}_{n'n''}, \tag{3.11}$$

where $\mathsf{A}_{n\,n'}\mathsf{f}_{n'}$ and $\mathsf{A}_{n\,n'}\mathsf{B}_{n'n''}$ are the inner products in the corresponding three-dimensional space.

In this notation, the single-reflection matrix (3.10) is represented by the matrix $\mathsf{R}(H)$ with a three-diagonal structure

$$\mathsf{R}_{n\,n-i}(H)=\begin{cases}H^i\bar{\mathsf{R}}_{n\,n-i}, & i=0,1,2,\\ 0, & \text{otherwise},\end{cases} \tag{3.12}$$

where

$$\bar{\mathsf{R}}_{n\,n'}(\sigma\sigma') = R_{n\,n'}(\sigma\sigma', m). \tag{3.13}$$

The arrays of source and image multipoles are represented by the arrays f and f^* with elements

$$\mathsf{f}_n(\sigma) = f_{n-\sigma\,m\,\sigma}, \quad \mathsf{f}^*_n(\sigma) = f^*_{n-\sigma\,m\,\sigma}. \tag{3.14}$$

Taking into account diagonal form (3.4) of the matrix R, relation (3.1) can be rewritten as

$$\mathsf{f}^* = \mathsf{R}(H)\mathsf{f}, \tag{3.15}$$

where, for simplicity, the dependence on the index m is not indicated.

D. Double-reflection identities

The reflection of the image f^* with respect to the plane of the original reflection returns the source force distribution f. This symmetry and Eq. (3.15) imply the double-reflection identity

$$\mathsf{R}(-H)\mathsf{R}(H) = \mathsf{I}^{\mathcal{S}}, \tag{3.16}$$

where $\mathsf{I}^{\mathcal{S}}$ is the unit matrix in the space $\mathcal{S}$. For the individual matrix components (3.12), Eq. (3.16) yields

$$\bar{\mathsf{R}}_{n\,n}\bar{\mathsf{R}}_{n\,n} = \mathsf{I}, \tag{3.17}$$

$$\bar{\mathsf{R}}_{n\,n-1}\bar{\mathsf{R}}_{n-1\,n-1} - \bar{\mathsf{R}}_{n\,n}\bar{\mathsf{R}}_{n\,n-1} = 0, \tag{3.18}$$

$$\bar{\mathsf{R}}_{n\,n-2}\bar{\mathsf{R}}_{n-2\,n-2} - \bar{\mathsf{R}}_{n\,n-1}\bar{\mathsf{R}}_{n-1\,n-2} + \bar{\mathsf{R}}_{n\,n}\bar{\mathsf{R}}_{n\,n-2} = 0, \tag{3.19}$$

$$\bar{\mathsf{R}}_{n\,n-2}\bar{\mathsf{R}}_{n-2\,n-3} - \bar{\mathsf{R}}_{n\,n-1}\bar{\mathsf{R}}_{n-1\,n-3} = 0, \tag{3.20}$$

$$\bar{\mathsf{R}}_{n\,n-2}\bar{\mathsf{R}}_{n-2\,n-4} = 0, \tag{3.21}$$

where I is the identity matrix in three dimensions corresponding to indices σ, σ' in the relation (3.13).

IV. TWO WALL SOLUTION

A. Double-reflection expansion

The double-reflection identities (3.17)–(3.21) can be used to derive explicit expressions for the multipolar moments of the image singularities $f^{(i\alpha)}_{lm\sigma}$ in the two-wall system. The problem is formulated in terms of multiple-reflection matrices $\mathsf{R}^{(i\alpha)}$, which are defined by the linear relation

$$\mathsf{f}^{(i\alpha)} = \mathsf{R}^{(i\alpha)}(h^{(\alpha)}, h)\mathsf{f}, \tag{4.1}$$

where the dependence of R on the wall position (2.1) and the separation of the walls are indicated. The array $\mathsf{f}^{(i\alpha)}$ is related to the elements $f^{(i\alpha)}_{lm\sigma}$ as implied by Eq. (3.14). By invariance with respect to the transformation $z \to -z$, the reflection matrices representing two families of images (a) and (b) are related,

$$\mathsf{R}^{(ib)}(H, h) = \mathsf{R}^{(ia)}(-H, -h). \tag{4.2}$$

In what follows, we thus consider only the family $\alpha = a$, and we set $H = h^{(a)}$.

The structure of matrices $\mathsf{R}^{(ia)}$ is analyzed by factorizing them into a double-reflection sequence. To this end, we introduce abbreviated notation

$$\mathsf{R}^{(2r\,a)}(H,h)=\mathsf{P}^{(r)} \tag{4.3}$$

and

$$\mathsf{R}^{(2r+1\,a)}(H,h)=\tilde{\mathsf{P}}^{(r)}. \tag{4.4}$$

Equations (3.2) and (4.1) imply that the odd-reflection matrices can be obtained from the relation

$$\tilde{\mathsf{P}}^{(r)}=\mathsf{R}(H_r)\mathsf{P}^{(r)}, \tag{4.5}$$

and the even-reflection matrices can be factorized

$$\mathsf{P}^{(r)}=\prod_{i=1}^{r}\mathsf{Q}(H_{i-1}). \tag{4.6}$$

Here

$$H_r=H+2rh \tag{4.7}$$

is the distance between the plane (a) and the image at the position $\mathbf{r}^{(2r\,a)}$, consistent with Eq. (3.3). The double-reflection matrix is defined

$$\mathsf{Q}(H)=\mathsf{R}(-H-h)\mathsf{R}(H), \tag{4.8}$$

and the order of the matrix product is $\Pi_{i=1}^{k}\mathsf{M}_i=\mathsf{M}_k\ldots\mathsf{M}_1$.

B. Simplified form of even-reflection matrices

Definitions (4.6) and (4.8) imply that the even-reflection matrix has the following structure,

$$\mathsf{P}^{(r)}_{n\,n'}=\sum_{\{n_j\}}\left[\alpha(\{n_j\})\prod_{k=1}^{2r}\bar{\mathsf{R}}_{n_k\,n_{k-1}}\right], \tag{4.9}$$

where the summation is over all sets $\{n_j\}$ of integer elements n_j $(j=0,1,\ldots,2r)$ that satisfy relations

$$n_0=n',\quad n_{2r}=n,\quad n_j-n_{j-1}=0,1,2. \tag{4.10}$$

The matrix $\mathsf{P}^{(r)}_{nn'}$ depends on H and h only through the coefficients

$$\alpha(\{n_j\})=\prod_{i=1}^{r}\left[(-h-H_{i-1})^{n_{2i}-n_{2i-1}}H_{i-1}^{n_{2i-1}-n_{2i-2}}\right], \tag{4.11}$$

which can be shown using Eq. (3.12).

Equation (4.9) can be considerably simplified by using the double-reflection identities (3.17)–(3.21). Accordingly, the commutation relations (3.18) and (3.20) are used to shift matrices $\bar{\mathsf{R}}_{k\,k-1}$ to the left; in the resulting formula products $\bar{\mathsf{R}}_{k\,k-1}\bar{\mathsf{R}}_{k-1\,k-2}$ are eliminated using (3.19), and products $\bar{\mathsf{R}}_{k\,k}\bar{\mathsf{R}}_{k\,k}$ and $\bar{\mathsf{R}}_{k\,k-2}\bar{\mathsf{R}}_{k-2\,k-4}$ are reduced using (3.17) and (3.21). Following this procedure, the even-reflection matrices $\mathsf{P}^{(r)}_{n\,n'}$ can be represented as linear combinations

$$\mathsf{P}^{(r)}_{n\,n-2s}=a(r,s)\mathsf{A}_{n\,n-2s}+b(r,s)\mathsf{B}_{n\,n-2s}, \tag{4.12}$$

$$\mathsf{P}^{(r)}_{n\,n-2s-1}=c(r,s)\mathsf{C}_{n\,n-2s-1}+d(r,s)\mathsf{D}_{n\,n-2s-1}, \tag{4.13}$$

where

$$\mathsf{A}_{n\,n}=\mathsf{B}_{n\,n}=\mathsf{I}, \tag{4.14}$$

$$\mathsf{A}_{n\,n-2s}=\prod_{i=n-2s}^{n-2}{}' \ (\bar{\mathsf{R}}_{i+2\,i}\bar{\mathsf{R}}_{i\,i}), \quad s=1,2,\ldots, \tag{4.15}$$

$$\mathsf{B}_{n\,n-2s}=\prod_{i=n-2s}^{n-2}{}' \ (\bar{\mathsf{R}}_{i+2\,i+2}\bar{\mathsf{R}}_{i+2\,i}), \quad s=1,2,\ldots, \tag{4.16}$$

$$\mathsf{C}_{n\,n-2s-1}=\bar{\mathsf{R}}_{n\,n}\bar{\mathsf{R}}_{n\,n-1}\mathsf{A}_{n-1\,n-1-2s}, \quad s=0,1,\ldots, \tag{4.17}$$

$$\mathsf{D}_{n\,n-2s-1}=\bar{\mathsf{R}}_{n\,n-1}\bar{\mathsf{R}}_{n-1\,n-3}\mathsf{B}_{n-3\,n-3-2(s-1)}, \quad s=1,2,\ldots, \tag{4.18}$$

and the remaining elements of $\mathsf{A}_{n\,n'}$, $\mathsf{B}_{n\,n'}$, $\mathsf{C}_{n\,n'}$, $\mathsf{D}_{n\,n'}$ are set equal to zero.

Explicit expressions for the matrices A, B are given in Appendix C; the matrices C and D can be obtained from these results using Eqs. (4.17) and (4.18) and the expressions listed in Appendix B. The scalar coefficients a, b, c, and d are evaluated in the following section.

C. Recurrence relations

To proceed with our analysis, we rewrite Eq. (4.6) in the form

$$\mathsf{P}^{(r+1)}=\mathsf{Q}(H_r)\mathsf{P}^{(r)}, \tag{4.19}$$

which in the component notation yields

$$\mathsf{P}^{(r+1)}_{n\,n-q}=\sum_{p=0}^{4}\mathsf{Q}_{n\,n-p}(H_r)\mathsf{P}^{(r)}_{n-p\,n-q}, \tag{4.20}$$

according to Eqs. (4.8) and (3.12). The elements of the matrix Q are simplified by using the double-reflection identities (3.17)–(3.21). It follows that

$$\mathsf{Q}_{n\,n}(H)=I, \tag{4.21}$$

$$\mathsf{Q}_{n\,n-1}(H)=-h\bar{\mathsf{R}}_{n\,n}\bar{\mathsf{R}}_{n\,n-1}, \tag{4.22}$$

$$\mathsf{Q}_{n\,n-2}(H)=h[(H+h)\bar{\mathsf{R}}_{n\,n-2}\bar{\mathsf{R}}_{n-2\,n-2}-H\bar{\mathsf{R}}_{n\,n}\bar{\mathsf{R}}_{n\,n-2}], \tag{4.23}$$

$$\mathsf{Q}_{n\,n-3}(H)=hH(H+h)\bar{\mathsf{R}}_{n\,n-1}\bar{\mathsf{R}}_{n-1\,n-3}, \tag{4.24}$$

$$\mathsf{Q}_{n\,n-4}(H)=0, \tag{4.25}$$

and all other elements $\mathsf{Q}_{n\,n'}$ vanish. Two pairs of coupled recurrence relations for the coefficients a, b, c, d in Eqs. (4.12) and (4.13) are derived from relation (4.20) by using the above expressions for Q and definitions (4.14)–(4.18),

$$\begin{aligned}a(r+1,s)=a(r,s)-h\,c(r,s-1)+h(h+H_r)[\delta_{s1}+(1-\delta_{s1})a(r,s-1)]\\ +h(h+H_r)\,H_r c(r,s-2),\end{aligned} \tag{4.26}$$

$$c(r+1,s)=c(r,s)-h[\delta_{s0}+(1-\delta_{s0})a(r,s)]-h\,H_r c(r,s-1), \tag{4.27}$$

and

$$b(r+1,s)=b(r,s)-h[d(r,s-1)-rh\delta_{s1}]-h\,H_r[\delta_{s1}+(1-\delta_{s1})b(r,s-1)], \tag{4.28}$$

$$\begin{aligned} d(r+1,s)=&d(r,s)-h(1-\delta_{s0})b(r,s)+h\,(h+H_r)[d(r,s-1)-rh\delta_{s1}]\\ &+h(h+H_r)\,H_r[\delta_{s1}+(1-\delta_{s1})b(r,s-1)]. \end{aligned} \tag{4.29}$$

The initial conditions for the above relations are obtained from the identity

$$\mathsf{P}^{(1)}_{nn'}=\mathsf{Q}_{n\,n'}(H) \tag{4.30}$$

and Eqs. (4.21)–(4.25), which imply

$$\begin{gathered} a(1,0)=1,\quad a(1,1)=h(h+H),\\ b(1,0)=0,\quad b(1,1)=-hH,\\ c(1,0)=-h,\quad c(1,1)=0,\\ d(1,0)=0,\quad d(1,1)=h(h+H)H, \end{gathered} \tag{4.31}$$

and

$$a(1,s)=b(1,s)=c(1,s)=d(1,s)=0,\qquad \text{for}\quad s>1\quad\text{or}\quad s<0. \tag{4.32}$$

We assumed here $a(1,0)=1$; however, because of Eq. (4.14), only $a(1,0)+b(0,1)=1$ is required.

The recurrence relations (4.26)–(4.29) with initial conditions (4.31) and (4.32) can be explicitly solved. It can be verified by induction that the solution is

$$a(r,s)=\frac{(-1)^{s+1}(r+s-1)!}{(2s)!(r-s)!}h^{2s-1}[2sH+(4sr-r-s)h], \tag{4.33}$$

$$b(r,s)=\frac{(-1)^{s}(r+s-1)!}{(2s)!(r-s)!}h^{2s-1}[2sH+(r-s)h](1-\delta_{s0}), \tag{4.34}$$

$$c(r,s)=\frac{(-1)^{s+1}(r+s)!}{(2s+1)!(r-s-1)!}h^{2s+1}, \tag{4.35}$$

$$\begin{aligned} d(r,s)=&\frac{(-1)^{s+1}(r+s-1)!}{(2s+1)!(r-s)!}h^{2s-1}\{2s(2s+1)[H^2+(2r-1)Hh]\\ &+(r-s)(r-s+4rs)h^2\}(1-\delta_{s0}). \end{aligned} \tag{4.36}$$

Using in the above equations the relation $k!=\infty$ for $k=-1,-2,\ldots$, we find

$$a(r,s)=b(r,s)=c(r,s)=d(r,s)=0,\qquad \text{for}\quad s<0,\quad\text{or}\quad s>r, \tag{4.37}$$

which is equivalent to

$$\mathsf{P}^{(r)}_{n\,n-q}=0\qquad \text{for}\quad q<0\quad\text{or}\quad q>2r+1. \tag{4.38}$$

The limit $r\to 0$ yields relations $a(0,0)+b(0,0)=1$ and $c(0,0)=d(0,0)=0$ corresponding to $\mathsf{P}^{(0)}=\mathsf{I}^{\mathcal{S}}$.

D. Odd-reflection matrices

Odd-reflection matrices $\tilde{\mathsf{P}}^{(r)}$ can be evaluated from our solution for the even-reflection problem. By inserting relations (4.12) and (4.13) into (4.5) and using identities (3.17)–(3.21), we find

$$\tilde{\mathsf{P}}^{(r)}_{n\,n-2s}=\tilde{a}(r,s)\bar{\mathsf{R}}_{n\,n}\mathsf{A}_{n\,n-2s}+\tilde{b}(r,s)\bar{\mathsf{R}}_{n\,n-2}\mathsf{B}_{n-2\,n-2s}, \tag{4.39}$$

$$\tilde{\mathsf{P}}^{(r)}_{n\,n-2s-1}=\tilde{c}(r,s)\bar{\mathsf{R}}_{n\,n-1}\mathsf{A}_{n-1\,n-2s-1}+\tilde{d}(r,s)\bar{\mathsf{R}}_{n\,n-1}\mathsf{B}_{n-1\,n-2s-1}, \tag{4.40}$$

where

$$\tilde{a}(r,s)=a(r,s)+H_r c(r,s-1), \tag{4.41}$$

$$\tilde{b}(r,s)=b(r,s)+H_r[d(r,s-1)+c(r,0)\,\delta_{s1}]+H_r^2[b(r,s-1)+\delta_{s1}], \tag{4.42}$$

$$\tilde{c}(r,s)=c(r,s)+H_r a(r,s)+H_r^2 c(r,s-1), \tag{4.43}$$

$$\tilde{d}(r,s)=d(r,s)+H_r b(r,s). \tag{4.44}$$

The above expressions, along with the results (4.12)–(4.18) and (4.33)–(4.36) for $\mathsf{P}^{(r)}$, provide a complete image representation of the flow reflected from the walls.

V. CONCLUSIONS

We have explored the structure of the multiple-reflection series for a Stokes-flow singularity in the space bounded by two parallel planar walls. Explicit expressions for the multipole moments of the images were derived using the symmetry properties of the reflection matrices to simplify the problem.

Accordingly, symmetries of the Lorentz's reflection operator were applied to show that the single-reflection matrix R has a lower-triangular/tri-diagonal structure in a properly chosen basis. The symmetry between the source and image singularities was shown to yield commutation identities for the elements of R. These identities were used to simplify the matrices $\mathsf{P}^{(r)}$ that represent even-order reflections of the flow field from the walls. In this form, $\mathsf{P}^{(r)}$ depends on the position of the source singularity and the reflection order r only through scalar prefactors, which are independent of the multipolar order l of the source. The prefactors were evaluated using recurrence formulas associated with subsequent even-order reflections.

Our image representation can be applied in investigations that involve the motion of particles suspended in a fluid confined between two parallel walls. Such problems include particle dynamics in highly asymmetric colloidal mixtures[22] and suspension flows in slit pores.[9–15] Single- and multi-particle mobility matrices for particles between two walls can be evaluated using the induced-force representation of the particles[23] and our image solution for the flow reflected from the walls. By this approach, the force multipoles induced on the particles are determined using appropriate multipole-expansion algorithms.[8,24] Our image representation can also be used to develop boundary-integral algorithms[25] for the motion of deformable drops between two walls.

Extensions of our work may include derivation of an image solution for a force multipole in a space bounded by two planar surfactant-free[1] or surfactant-covered[5] fluid-fluid interfaces. Such solutions would be useful in investigations of the dynamics of colloidal-particle- or micelle-stabilized thin liquid films.[26,27]

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APPENDIX A: FUNDAMENTAL SET OF VELOCITY FIELDS

Following definitions given in Ref. 20, we list expressions for the functions $\mathbf{V}_{lm\sigma}(\theta,\phi)$ that characterize angular dependence of the velocity fields (2.3):

$$\mathbf{V}_{lm0}=\frac{1}{(2l+1)^2}\left[\frac{l+1}{l(2l-1)}\hat{\mathbf{A}}_{lm}-\frac{1}{2}\hat{\mathbf{B}}_{lm}\right], \tag{A1}$$

$$\mathbf{V}_{lm1}=\frac{\mathrm{i}}{l(l+1)(2l+1)}\hat{\mathbf{C}}_{lm}, \tag{A2}$$

$$\mathbf{V}_{lm2}=\frac{l}{(l+1)(2l+1)^2(2l+3)}\hat{\mathbf{B}}_{lm}. \tag{A3}$$

Here

$$\hat{\mathbf{A}}_{lm}=r^{-l+1}\boldsymbol{\nabla}(r^l\hat{Y}_{lm}),\quad \hat{\mathbf{B}}_{lm}=r^{l+2}\boldsymbol{\nabla}(r^{-l-1}\hat{Y}_{lm}),\quad \hat{\mathbf{C}}_{lm}=\hat{\mathbf{A}}_{lm}\times\mathbf{e}_r \tag{A4}$$

(with $r=|\mathbf{r}|$ and $\mathbf{e}_r=\mathbf{r}/r$) are the unnormalized vector spherical harmonics, where

$$\hat{Y}_{lm}=(-1)^m P_l^m(\cos\theta)e^{im\phi} \tag{A5}$$

are the unnormalized scalar spherical harmonics.

APPENDIX B: ELEMENTS OF REFLECTION MATRIX

Here we list expressions for nonzero elements of the reflection matrix $\bar{\mathsf{R}}_{n\,n'}(\sigma\sigma')$, defined by Eqs. (3.10) and (3.13),

$$\bar{\mathsf{R}}_{n\,n}(00)=(-1)^{(n+m+1)}\left[1+\frac{2(n-2)(n+m)(n-m)}{n(2n-1)}\right], \tag{B1}$$

$$\bar{\mathsf{R}}_{n\,n}(01)=(-1)^{(n+m)}\frac{4m(n+m)}{n(n-1)}, \tag{B2}$$

$$\bar{\mathsf{R}}_{n\,n}(02)=(-1)^{(n+m+1)}\frac{4(n-2)(n+m)(n+m-1)}{(n-1)(2n-1)(2n-3)}, \tag{B3}$$

$$\bar{\mathsf{R}}_{n\,n}(10)=(-1)^{(n+m+1)}\frac{2(n-2)m(n-m)}{n(2n-1)}, \tag{B4}$$

$$\bar{\mathsf{R}}_{n\,n}(11)=(-1)^{(n+m+1)}\left[1-\frac{4m^2}{(n-1)n}\right], \tag{B5}$$

$$\bar{\mathsf{R}}_{n\,n}(12)=(-1)^{(n+m+1)}\frac{4(n-2)m(n-1+m)}{(n-1)(2n-3)(2n-1)}, \tag{B6}$$

$$\bar{\mathsf{R}}_{n\,n}(20)=(-1)^{(n+m)}\frac{(n+1)(2n-3)(n-m-1)(n-m)}{n(2n-1)}, \tag{B7}$$

$$\bar{\mathsf{R}}_{n\,n}(21)=(-1)^{(n+m+1)}\frac{2m(n+1)(2n-3)(n-m-1)}{(n-2)(n-1)n}, \tag{B8}$$

$$\bar{\mathsf{R}}_{n\,n}(22)=(-1)^{(n+m+1)}\left[1-\frac{2(n+1)(n-m-1)(n+m-1)}{(n-1)(2n-1)}\right], \tag{B9}$$

$$\bar{\mathsf{R}}_{n\,n-1}(00)=(-1)^{(n+m+1)}2(n+m), \tag{B10}$$

$$\bar{\mathsf{R}}_{n\,n-1}(10)=(-1)^{(n+m+1)}2m, \tag{B11}$$

$$\bar{\mathsf{R}}_{n\,n-1}(20)=(-1)^{(n+m)}4n(n-m-1), \tag{B12}$$

$$\bar{\mathsf{R}}_{n\,n-1}(21)=(-1)^{(n+m+1)}\frac{2m(2n-3)(2n-1)}{(n-2)^2}, \tag{B13}$$

$$\bar{\mathsf{R}}_{n\,n-1}(22)=(-1)^{(n+m)}\frac{2(n-3)(n-1)(2n-1)(n+m-2)}{(2n-5)(n-2)^2}, \tag{B14}$$

$$\bar{\mathsf{R}}_{n\,n-2}(20)=(-1)^{(n+m)}\frac{(n-1)(2n-3)(2n-1)}{n-2}. \tag{B15}$$

In the above expressions we set

$$\bar{\mathsf{R}}_{n\,n'}(\sigma\sigma')=0 \quad \text{for} \quad n'-\sigma'<1. \tag{B16}$$

Equations (B1)–(B15) are equivalent to Eqs. (A20) in Ref. 4.

APPENDIX C: PRODUCTS OF MATRICES $\bar{\mathsf{R}}$

Here we list expressions for nonzero elements of matrices A and B, defined by Eqs. (4.15) and (4.16),

$$\mathsf{A}_{n\,n-2s}=f(n,s)\mathsf{A}'_{n\,n-2s}, \tag{C1}$$

$$\mathsf{B}_{n\,n-2s}=f(n,s)\mathsf{B}'_{n\,n-2s}, \tag{C2}$$

where $s\geqslant 1$, and

$$f(n,s)=(-4)^s\frac{(n-1)(2n-1)(2n-3)}{n-2}\,\frac{(n+m-2)!}{(n+m-2s)!}, \tag{C3}$$

$$\mathsf{A}'_{n\,n-2s}(20)=\frac{1}{4}\left[1+\frac{2(n-2s-2)(n-2s-m)(n-2s+m)}{(2n-4s-1)(n-2s)}\right], \tag{C4}$$

$$\mathsf{A}'_{n\,n-2s}(21)=-\frac{m(m+n-2s)}{(n-2s-1)(n-2s)}, \tag{C5}$$

$$\mathsf{A}'_{n\,n-2s}(22)=\frac{(m+n-2s-1)(m+n-2s)(n-2s-2)}{(2n-4s-3)(2n-4s-1)(n-2s-1)}, \tag{C6}$$

$$\mathsf{B}'_{n\,n-2s}(00)=\frac{(n-2)(m+n)(m+n-1)}{(n-1)(2n-1)(2n-3)}, \tag{C7}$$

$$\mathsf{B}'_{n\,n-2s}(10)=\frac{(n-2)m(m+n-1)}{(n-1)(2n-1)(2n-3)}, \tag{C8}$$

$$\mathsf{B}'_{n\,n-2s}(20)=\frac{1}{4}\left[1-\frac{2(n+1)(n-m-1)(n+m-1)}{(n-1)(2n-1)}\right]. \tag{C9}$$

In the above expressions we set

$$\mathsf{A}_{n\,n'}(\sigma\sigma')=\mathsf{B}_{n\,n'}(\sigma\sigma')=0 \quad \text{for} \quad n'-\sigma'<1. \tag{C10}$$

Relations (C1)–(C10) can be verified by induction using definitions (4.15) and (4.16) and the expressions in Appendix B.

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