# Exponents, symmetry groups and classification of operator fractional Brownian motions [*†‡§]

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**Abstract**

Operator fractional Brownian motions (OFBMs) are zero mean, operator self-similar (o.s.s.), Gaussian processes with stationary increments. They generalize univariate fractional Brownian motions to the multivariate context. It is well-known that the so-called symmetry group of an o.s.s. process is conjugate to subgroups of the orthogonal group. Moreover, by a celebrated result of Hudson and Mason, the set of all exponents of an operator self-similar process can be related to the tangent space of its symmetry group.

In this paper, we revisit and study both the symmetry groups and exponent sets for the class of OFBMs based on their spectral domain integral representations. A general description of the symmetry groups of OFBMs in terms of subsets of centralizers of the spectral domain parameters is provided. OFBMs with symmetry groups of maximal and minimal types are studied in any dimension. In particular, it is shown that OFBMs have minimal symmetry groups (as thus, unique exponents) in general, in the topological sense. Finer classification results of OFBMs, based on the explicit construction of their symmetry groups, are given in the lower dimensions 2 and 3. It is also shown that the parametrization of spectral domain integral representations are, in a suitable sense, not affected by the multiplicity of exponents, whereas the same is not true for time domain integral representations.

## 1 Introduction

This work is about the class of operator fractional Brownian motions (OFBMs). Denoted by $B_H = \{B_H(t)\}_{t\in\mathbb{R}} = \{(B_{H,1}(t),\dots,B_{H,n}(t))' \in \mathbb{R}^n, t \in \mathbb{R}\}$, these are multivariate zero mean Gaussian processes with stationary increments which are operator self-similar (o.s.s.) with a matrix exponent $H$. Operator self-similarity means that, for any $c > 0$,

$$\{B_H(ct)\}_{t\in\mathbb{R}} \stackrel{\mathcal{L}}{=} \{c^H B_H(t)\}_{t\in\mathbb{R}}, \tag{1.1}$$

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where $=_{\mathcal{L}}$ stands for the equality of finite dimensional distributions and $c^H := e^{H \ln c} := \sum_{k=0}^{\infty} H^k (\ln c)^k / k!$. It is also assumed that OFBMs are proper, in the sense that the support of the distribution of $B_H(t)$ is $\mathbb{R}^n$ for every $t \in \mathbb{R}$. OFBMs play an important role in the analysis of multivariate time series, analogous to that of the usual fractional Brownian motion (FBM) in the univariate context. They have been studied more systematically by Mason and Xiao (2002), Bahadaron, Benassi and Dębicki (2003), Lavancier, Philippe and Surgailis (2009), Didier and Pipiras (2010), and others. Regarding o.s.s. processes in general, see Hudson and Mason (1982), Laha and Rohatgi (1981), Sato (1991), Maejima and Mason (1994), Maejima (1996, 1998), Meerschaert and Scheffler (1999), Section 11 in Meerschaert and Scheffler (2001), Chapter 9 in Embrechts and Majima (2002), Becker-Kern and Pap (2008). For related work on operator stable measures, see, for instance, Sharpe (1969), Jurek and Mason (1993), Meerschaert and Veeh (1993, 1995), Hudson and Mason (1981), among others.

In particular, Didier and Pipiras (2010) showed that, under the mild assumption

$$0 < \Re(h_k) < 1, \quad k = 1, \ldots, n, \tag{1.2}$$

on the eigenvalues $h_k$ of the matrix exponent $H$, any OFBM $B_H$ admits the so-called integral representation in the spectral domain,

$$\{B_H(t)\}_{t \in \mathbb{R}} \stackrel{\mathcal{L}}{=} \Big\{ \int_{\mathbb{R}} \frac{e^{itx} - 1}{ix} (x_+^{-(H - \frac{1}{2}I)} A + x_-^{-(H - \frac{1}{2}I)} \overline{A}) \widetilde{B}(dx) \Big\}_{t \in \mathbb{R}}. \tag{1.3}$$

Here, $x_{\pm} = \max\{\pm x, 0\}$,

$$A = A_1 + iA_2 \tag{1.4}$$

is a complex-valued matrix with real-valued $A_1, A_2$, $\overline{A}$ indicates the complex conjugate of $A$, $\widetilde{B}(x) = \widetilde{B}_1(x) + i\widetilde{B}_2(x)$ is a complex-valued multivariate Brownian motion satisfying $\widetilde{B}_1(-x) = \widetilde{B}_1(x)$, $\widetilde{B}_2(-x) = -\widetilde{B}_2(x)$, and $\widetilde{B}_1$ and $\widetilde{B}_2$ are independent with induced random measure $\widetilde{B}(dx)$ satisfying $E\widetilde{B}(dx)\widetilde{B}(dx)^* = dx$. Thus, according to (1.3), OFBMs are characterized (parametrized) by the matrices $H$ and $A$.

In this work, we continue the systematic study of OFBMs started in Didier and Pipiras (2010). We now tackle the issues of the symmetry structure of OFBMs and of the non-uniqueness (multiplicity) of the exponents $H$, which, in our view, are essentially unexplored. Such issues are strongly connected. Since the fundamental work of Hudson and Mason (1982), it is well known that one given o.s.s. process $X$ may have multiple exponents. More specifically, if we denote the set of exponents of $X$ by $\mathcal{E}(X)$, we have that

$$\mathcal{E}(X) = H + T(G_X), \tag{1.5}$$

where $H$ is any particular exponent of the process $X$. Here,

$$G_X = \Big\{ C \in GL(n) : \{X(t)\}_{t \in \mathbb{R}} \stackrel{\mathcal{L}}{=} \{CX(t)\}_{t \in \mathbb{R}} \Big\} \tag{1.6}$$

is the so-called symmetry group of the process $X$ (where $GL(n)$ is the multiplicative group of invertible matrices), and

$$T(G_X) = \Big\{ C : C = \lim_{n \to \infty} \frac{C_n - I}{d_n}, \quad \text{for some } \{C_n\} \subseteq G_X,\ 0 < d_n \to 0 \Big\} \tag{1.7}$$

is the tangent space of the symmetry group $G_X$. By a result for compact groups (e.g., Hoffman and Morris (1998), p. 49, or Hudson and Mason (1982), p. 285), it is known that

$$G_X = W \mathcal{O}_0 W^{-1} \tag{1.8}$$

for some positive definite matrix $W$ and some subgroup $\mathcal{O}_0$ of the orthogonal group. As a consequence, the knowledge about (1.5) is subordinated to that about the symmetry group $G_X$ of $X$. For example, the exponent is unique for the process $X$ if and only if the symmetry group $G_X$ is finite.

The description and study of symmetry groups beyond the decomposition (1.8) is a reputedly difficult and interesting problem (see, for instance, Billingsley (1966), p. 176, and Jurek and Mason (1993), p. 60, both in the context of random vectors; see also Meerschaert and Veeh (1993, 1995)). In this paper, we take up and provide some answers for this challenging problem in the context of OFBMs. The main goal of this paper is two-fold: to study the symmetry groups of OFBMs in as much detail as possible, and based on this, to closely examine (1.5) for OFBMs $X = B_H$. We emphasize again that, to the best of our knowledge, this is the first work where symmetry groups are examined for any large class of o.s.s. processes (e.g., for the notion of symmetry groups of Markov processes, see Liao (1992) and references therein). Indeed, since its publication, the scope of the work of Hudson and Mason (1982) appears to have remained only of general nature, the same being true for the main result (1.5).

The integral representation (1.3) provides a natural and probably the only means to consider (almost) the whole class of OFBMs. Section 3 is dedicated to the reinterpretation and explicit representation of symmetry-related constructs in terms of the spectral parametrization $H$, $A$. One of our main results provides a decomposition of the symmetry groups of OFBMs into the intersection of (subsets of) centralizers, i.e., sets of matrices that commute with a given matrix. For example, in the case of time reversible OFBMs (corresponding to the case when $AA^* = \overline{AA^*}$), we show that the symmetry group $G_{B_H}$ is conjugate to

$$\bigcap_{x>0} G(\Pi_x). \tag{1.9}$$

Here, $G(\Pi)$ denotes the centralizer of a matrix $\Pi$ in the group $O(n)$ of orthogonal matrices, i.e.,

$$G(\Pi) = \{O \in O(n) : O\Pi = \Pi O\}, \tag{1.10}$$

and the matrix-valued function $\Pi_x$ has the frequency $x$ as the argument and is parametrized by $H$ and $A$. Moreover, which is key for many technical results in this paper, we actually express the positive definite conjugacy matrix $W$ in (1.8) in terms of the spectral parametrization. This is a substantial improvement over previous works on operator self-similarity, where only the existence of such conjugacy is obtained, e.g., as in (1.8).

In view of (1.9) and (1.10), it is clear that the symmetry structure of OFBM is rooted in centralizers. The characterization of the commutativity of matrices is a well-studied algebraic problem (e.g., MacDuffee (1946), Taussky (1953), Gantmacher (1959), Suprunenko and Tyshkevich (1968), Lax (2007)). We apply the available techniques in a variety of ways to provide a detailed study of the symmetry groups and the associated tangent spaces (Sections 4 and 5), as well as of the consequences of the non-uniqueness of the parametrization for integral representations (Section 6).

Our study of the symmetry structures of OFBMs is carried out from two perspectives: first, by looking at the extremal cases, i.e., maximal and minimal symmetry for arbitrary dimension, and second, by conveying a full description of *all* symmetry groups in the lower dimensions $n = 2$ and $n = 3$.

Section 4 is dedicated to the first perspective. We completely characterize OFBMs with maximal symmetry, i.e., those whose symmetry groups are conjugate to $O(n)$. We establish the general form of their covariance function and of their spectral parametrization. However, as intuitively clear, maximal symmetry corresponds to a strict subset of the parameter space of OFBMs. In view of this, one can naturally ask what the most typical symmetry structure for OFBMs is, in a suitable sense. A related question is whether the multiplicity of exponents (and, thus, the non-identifiability of the parametrization) is a general phenomenon. Section 4 contains our answer to both questions, which is, indeed, one of our main results. We prove that, in the topological sense, OFBMs with minimal symmetry groups (i.e., $\{I, -I\}$) form the *largest* class within all OFBMs. As a consequence, in the same sense, OFBMs generally have *unique* exponents. To establish this result, in our analysis of the centralizers $G(\Pi_x)$, we bypass the need to deal with the major complexity of the eigenspace structure of the function $\Pi_x$ by looking at its behavior at the origin of the Lie group (i.e., as $x \to 1$), where a great deal of information about $\Pi_x$ is available through the celebrated Baker-Campbell-Hausdorff formula.

Section 5 contains a full description of the symmetry structure of low-dimensional OFBMs, namely, for dimensions $n = 2$ and $n = 3$. We provide a classification of OFBMs based on their symmetry groups. For example, when $n = 2$, the symmetry group of a general OFBM can be, up to a conjugacy, of only one of the following types:

$(i)$ minimal: $\{I, -I\}$;

$(ii)$ trivial: $\{I, -I, R, -R\}$, where $R$ is a reflection matrix;

$(iii)$ rotational: $SO(2)$ (the group of rotation matrices);

$(iv)$ maximal: $O(2)$.

Such classification of types for $n = 2$ stands in contrast with the situation with random vectors, for which $SO(n)$ cannot be a symmetry group (Billingsley (1966)). Nevertheless, we show that the latter statement is *almost* true for OFBMs, since $SO(n)$ cannot be a symmetry group if $n \geq 3$. In both $n = 2$ and $n = 3$, we provide examples of OFBMs in all identified classes, and also discuss the structure of the resulting exponent sets $\mathcal{E}(B_H)$.

In Section 6, we examine the consequences of non-identifiability for integral representations of OFBMs. We show that the multiplicity of the exponents $H$ does *not* affect the parameter $A$ in (1.3) in the sense that the latter can be chosen the same for any of the exponents. Intriguingly, this turns out *not* to be the case for the parameters in the time domain representation of OFBMs, and points to one advantage of spectral domain representations.

In summary, the structure of the paper is as follows. Some preliminary remarks and notation can be found in Section 2. Section 3 concerns structural results on the symmetry groups of OFBMs. OFBMs with maximal and minimal symmetry groups are studied in Section 4. The classification of OFBMs according to their symmetry groups in the lower dimensions $n = 2$ and $n = 3$ can be found in Section 5. Section 6 contains results on the consequences of the non-uniqueness of the parametrization for integral representations. The appendix contains several auxiliary facts for the reader's convenience.

# 2 Preliminaries

## 2.1 Notation

We shall use throughout the paper the following notation for finite-dimensional operators (matrices). All with respect to the field $\mathbb{R}$, $M(n)$ or $M(n, \mathbb{R})$ is the vector space of all $n \times n$ operators (endomorphisms), $GL(n)$ or $GL(n, \mathbb{R})$ is the general linear group (invertible operators, or automorphisms), $O(n)$ is the orthogonal group of operators $O$ such that $OO^* = I = O^*O$ (i.e., the adjoint operator is the inverse), $SO(n) \subseteq O(n)$ is the special orthogonal group of operators (rotations) with determinant equal to 1, and $so(n)$ is the vector space of skew-symmetric operators (i.e., $A^* = -A$). Similarly, $M(m, n, \mathbb{R})$ is the space of $m \times n$ real matrices. The notation will indicate the change to the field $\mathbb{C}$. For instance, $M(n, \mathbb{C})$ is the vector space of complex endomorphisms. Whenever it is said that $A \in M(n)$ has a complex eigenvalue or eigenspace, one is considering the operator embedding $M(n) \hookrightarrow M(n, \mathbb{C})$. $U(n)$ is the group of unitary matrices, i.e., $UU^* = I = U^*U$. We will say that two endomorphisms $A, B \in M(n)$ are *conjugate* (or similar) when there exists $P \in GL(n)$ such that $A = PBP^{-1}$. In this case, $P$ is called a *conjugacy.* The expression $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$ denotes the operator whose matrix expression has the values $\lambda_1, \dots, \lambda_n$ on the diagonal and zeros elsewhere. We make no conceptual distinction between characteristic roots and eigenvalues. We also write $S^{n-1} := \{v \in \mathbb{R}^n : \|v\| = 1\}$. $\mathbf{0}$ represents a matrix of zeroes of suitable dimension. Unless otherwise stated, we consider the so-called spectral matrix norm $\|\cdot\|$, i.e., $\|A\|$ is the square root of the largest eigenvalue of $A^*A$. For $\{A_n\}_{n\in\mathbb{N}}$, $A \in M(n, \mathbb{C})$, we write $A_n \to A$ when $\|A_n - A\| \to 0$.

Throughout the paper, we set

$$D = H - \frac{1}{2}, \tag{2.1}$$

for an operator exponent $H$. We shall also work with the real part $\Re(AA^*) = A_1A_1^* + A_2A_2^*$ and the imaginary part $\Im(AA^*) = A_2A_1^* - A_1A_2^*$ of $AA^*$. For the real part, in particular, we will use the decomposition

$$\Re(AA^*) = S_R\Lambda_R^2 S_R^* = W^2, \tag{2.2}$$

with an orthogonal $S_R$, a diagonal $\Lambda_R$ and a positive (semi-)definite

$$W = S_R \Lambda_R S_R^*. \tag{2.3}$$

We shall use the assumption that

$$\Re(AA^*) \text{ has full rank}, \tag{2.4}$$

in which case $\Lambda_R$ in (2.2) has the inverse $\Lambda_R^{-1}$. As shown in Didier and Pipiras (2010), the condition (2.4) is sufficient (though not necessary) for the integral in (1.3) to be proper and hence to define an OFBM.

All through the paper, we assume $n \geq 2$.

## 2.2 Remarks on the multiplicity of matrix exponents

In this section, we make a few remarks to a reader less familiar with the subject of this work. It may appear a bit surprising that an o.s.s. process may have multiple exponents, as formalized in (1.5). This can be understood from at least two inter-related perspectives: the properties of operator (matrix) exponents and the distributional properties of o.s.s. processes. From the first perspective, consider for example matrices of the form

$$L_s = \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix} \in so(2), \tag{2.5}$$

where $s \in \mathbb{R}$. Being normal, these matrices can be diagonalized as $L_s = U_2\Lambda_s U_2^*$, where $U_2 \in U(2)$ and $\Lambda_s = \mathrm{diag}(is, -is)$. In particular, $\exp(L_{2\pi k}) = I$, $k \in \mathbb{Z}$, since $e^{i2\pi k} = 1$. Since $L_s$ and $L_{s'}$ commute for any $s, s' \in \mathbb{R}$, this yields

$$\exp(L_s) = \exp(L_{2\pi k}) \exp(L_s) = \exp(L_{2\pi k} + L_s), \tag{2.6}$$

and shows the potential non-uniqueness of operator exponents stemming from purely operator (matrix) properties. Note also that the situation here is quite different from the 1-dimensional case: in the latter, the same is possible but only with complex exponents, whereas here the matrices $L_{2\pi k}$ have purely real entries.

From the perspective of distributional properties, we can illustrate several ideas through the following simple example. The OFBMs in this example will appear again in Section 4 below.

**Example 2.1** (Single parameter OFBM) Consider an OFBM $B_H$ with covariance function $EB_H(t)B_H(s)^* =: \Gamma(t,s) = \Gamma_h(t,s)I$, where $\Gamma_h(t,s)$ is the covariance function of a standard univariate FBM with parameter $h \in (0,1)$. This process is o.s.s. with exponent $H = hI$, and will be called a single parameter OFBM. Since $B_H$ is Gaussian, $O \in G_{B_H}$ if and only if $O\Gamma(t,s)O^* = \Gamma(t,s)$. In the case of a single parameter OFBM, this is equivalent to $OO^* = I$ or, since $O$ has an inverse ($B_H$ is assumed proper), $OO^* = O^*O = I$. In other words, $G_{B_H} = O(n)$ and

$$\mathcal{E}(B_H) = H + T(O(n)) = H + so(n).$$

Thus, a single parameter OFBM has multiple exponents. From another angle, for a given $c > 0$ and $L \in so(n)$, we have $L\log(c) \in so(n)$ and hence $\exp(L\log(c)) = c^L \in O(n) = G_{B_H}$. Then,

$$\{B_H(ct)\}_{t\in\mathbb{R}} \stackrel{\mathcal{L}}{=} \{c^H B_H(t)\}_{t\in\mathbb{R}} \stackrel{\mathcal{L}}{=} \{c^H c^L B_H(t)\}_{t\in\mathbb{R}} \stackrel{\mathcal{L}}{=} \{c^{H+L} B_H(t)\}_{t\in\mathbb{R}},$$

which also shows that the exponents are not unique.

For later use, we also note that an equivalent way to define a single parameter OFBM is to say that it has the spectral representation

$$\{B_H(t)\}_{t\in\mathbb{R}} \stackrel{\mathcal{L}}{=} \Big\{C\int_{\mathbb{R}} \frac{e^{itx}-1}{ix}|x|^{-(h-\frac{1}{2})}\widetilde{B}(dx)\Big\}_{t\in\mathbb{R}}, \tag{2.7}$$

where $C$ is an appropriate normalizing constant and $\widetilde{B}(dx)$ is as in (1.3).

### 2.3 Basics of matrix commutativity

We now recap some key facts and results about matrix commutativity that are repeatedly used in the paper. To put it shortly, two matrices $A, B \in M(n,\mathbb{C})$ commute if and only if they share a common basis of generalized eigenvectors (see Lax (2007), p. 74). This means that there exists a matrix $P \in GL(n,\mathbb{C})$ such that we can write $A = PJ_AP^{-1}$ and $B = PJ_BP^{-1}$, where $J_A$ and $J_B$ are in Jordan canonical form. In particular, if $A$, $B$ are diagonalizable, then they must share a basis of eigenvectors. When, for example, $A = I$, we can interpret that $A$ commutes with any $B = PJ_BP^{-1} \in M(n,\mathbb{C})$ because for (any) $P \in GL(n,\mathbb{C})$, $A = PIP^{-1}$.

A related issue is that of the characterization of the set of all matrices that commute with a given matrix $A$, the so-called centralizer $\mathcal{C}(A)$. In particular, one is often interested in constructing the latter based on the Jordan decomposition of $A$.

Before enunciating the main theorem on $\mathcal{C}(A)$, we look at an example adapted from Gantmacher (1959).

**Example 2.2** Assume the matrix $A \in M(10,\mathbb{C})$, with Jordan representation $A = PJ_AP^{-1}$, has the elementary divisors (i.e., the characteristic polynomials of the Jordan blocks)

$$(\lambda-\lambda_1)^3, (\lambda-\lambda_1)^2, (\lambda-\lambda_2)^2, (\lambda-\lambda_3), (\lambda-\lambda_3), (\lambda-\lambda_3), \tag{2.8}$$

where the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are pairwise distinct. Then, $\mathcal{C}(A)$ is made up of matrices of the form $X = P\widetilde{X}P^{-1}$, where

$$M(10, \mathbb{C}) \ni \widetilde{X} = \left( \begin{array}{ccc|cc|cc|c|c|c} a & & & & & & & & & \\ b & a & & f & & & & & & \\ c & b & a & g & f & & & & & \\ \hline d & & & h & & & & & & \\ e & d & & i & h & & & & & \\ \hline & & & & & j & & & & \\ & & & & & k & j & & & \\ \hline & & & & & & & l & m & n \\ \hline & & & & & & & o & p & q \\ \hline & & & & & & & r & s & t \end{array} \right). \tag{2.9}$$

and the blocks on the diagonal correspond to the Jordan blocks of $J_A$.

We now turn to the structure of the blocks for the general case, and (2.9) serves as an illustration of the latter. We say that a matrix $X \in M(p_\alpha, q_\beta, \mathbb{C})$ has regular lower triangular form (e.g., as each of the blocks in (2.9) with letters $a$ through $j$) if it can be written as

$$X_{\alpha\beta} = \begin{cases} (T_{p_\alpha}, \mathbf{0}), & \text{if } p_\alpha \leq q_\beta, \\ (\mathbf{0}', T'_{q_\beta})', & \text{if } p_\alpha > q_\beta, \end{cases} \tag{2.10}$$

where $T_{p_\alpha} \in M(p_\alpha, \mathbb{C})$ is a Toeplitz lower triangular matrix. Also denote by $N_{p_\alpha} \in M(p_\alpha, \mathbb{C})$ the nilpotent matrix

$$N_{p_\alpha} = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

The next theorem characterizes $\mathcal{C}(A)$. The proof can be found in Gantmacher (1959), p. 219 (see also pp. 220-224).

**Theorem 2.1** *Let $A \in M(n, \mathbb{C})$, where $A = PJ_AP^{-1}$ and $J_A$ is in Jordan canonical form, i.e.,*

$$J_A = \operatorname{diag}(\lambda_1 I_{p_1} + N_{p_1}, \ldots, \lambda_u I_{p_u} + N_{p_u})$$

*with not necessarily distinct eigenvalues $\lambda_1, \ldots, \lambda_u$. Then, the general solution to the equation $AX = XA$ is given by the formula $X = PX_{J_A}P^{-1}$, where $X_{J_A}$ is the general solution to the equation $J_AX_{J_A} = X_{J_A}J_A$. Here, $X_{J_A}$ can be decomposed into blocks $X_{\alpha\beta} \in M(p_\alpha, q_\beta, \mathbb{C})$, $\alpha, \beta = 1, \ldots, u$, where*

$$X_{\alpha\beta} = \begin{cases} \mathbf{0}, & \text{if } \lambda_\alpha \neq \lambda_\beta, \\ \text{as in (2.10)}, & \text{if } \lambda_\alpha = \lambda_\beta. \end{cases}$$

In view of Theorem 2.1, it is intuitively clear that, if a matrix $\Gamma$ commutes with two matrices $A$ and $B$ which exhibit completely different sets of eigenvectors, then $\Gamma$ can only be a multiple of the identity. This is accurately stated in the next lemma, which is used several times in the paper.

**Lemma 2.1** *Let $A, B, \Gamma \in M(n, \mathbb{C})$, where both $A$ and $B$ have pairwise different eigenvalues, and $A$ and $B$ do not have any eigenvector in common. If $\Gamma$ commutes with both $A$ and $B$, then $\Gamma = \lambda I$, $\lambda \in \mathbb{C}$.*

The proof of Lemma 2.1 can be found in Appendix A, together with some additional results on matrix commutativity.

## 3 Symmetry groups of OFBMs

Consider an OFBM $B_H$ with the spectral representation (1.3). In this section, we provide some structural results on the nature of the symmetry group $G_{B_H}$ (see (1.6)). In particular, we explicitly express it as an intersection of subsets of centralizers.

For notational simplicity, denote $G_{B_H}$ by $G_H$. Since OFBMs are Gaussian and two Gaussian processes with stationary increments have the same law when (and only when) their spectral densities are equal a.e., we obtain that

$$\begin{aligned} G_H &= \{C \in GL(n) : EB_H(t)B_H(s)^* = E(CB_H(t))(CB_H(s))^*,\ s,t \in \mathbb{R}\} \\ &= \{C \in GL(n) : (x_+^{-D}A + x_-^{-D}\overline{A})(x_+^{-D}A + x_-^{-D}\overline{A})^* \\ &\qquad\qquad = C(x_+^{-D}A + x_-^{-D}\overline{A})(x_+^{-D}A + x_-^{-D}\overline{A})^*C^*,\ x \in \mathbb{R}\} \\ &= \{C \in GL(n) : x^{-D}AA^*x^{-D^*} = Cx^{-D}AA^*x^{-D^*}C^*,\ x > 0\} \\ &= G_{H,1} \bigcap G_{H,2}, \end{aligned} \tag{3.1}$$

where

$$G_{H,1} = \{C \in GL(n) : x^{-D}\Re(AA^*)x^{-D^*} = Cx^{-D}\Re(AA^*)x^{-D^*}C^*,\ x > 0\}, \tag{3.2}$$

$$G_{H,2} = \{C \in GL(n) : x^{-D}\Im(AA^*)x^{-D^*} = Cx^{-D}\Im(AA^*)x^{-D^*}C^*,\ x > 0\}. \tag{3.3}$$

Consider first the set $G_{H,1}$. Using the decomposition (2.2) and working under the assumption (2.4), we have that

$$\begin{aligned} G_{H,1} &= \{C \in GL(n) : x^{-D}S_R\Lambda_R^2S_R^*x^{-D^*} = Cx^{-D}S_R\Lambda_R^2S_R^*x^{-D^*}C^*,\ x > 0\} \\ &= \{C \in GL(n) : (\Lambda_R^{-1}S_R^*x^DCx^{-D}S_R\Lambda_R)(\Lambda_R^{-1}S_R^*x^DCx^{-D}S_R\Lambda_R)^* = I,\ x > 0\} \\ &= \{C \in GL(n) : \Lambda_R^{-1}S_R^*x^DCx^{-D}S_R\Lambda_R \in O(n),\ x > 0\}. \end{aligned} \tag{3.4}$$

Taking $x = 1$ and using the fact that $S_R$ is orthogonal, $C \in G_{H,1}$ necessarily has the form

$$C = S_R\Lambda_RS_R^*OS_R\Lambda_R^{-1}S_R^* = WOW^{-1} \tag{3.5}$$

with $O \in O(n)$ (see also Remark 3.1 below). Substituting (3.5) back into (3.4), we can now express $G_{H,1}$ as

$$\begin{aligned} G_{H,1} &= W\{O \in O(n) : \\ &\qquad O(W^{-1}x^{-D}\Re(AA^*)x^{-D^*}W^{-1}) = (W^{-1}x^{-D}\Re(AA^*)x^{-D^*}W^{-1})O,\ x > 0\}W^{-1} \\ &= W\bigcap_{x>0} G(\Pi_x)W^{-1}, \end{aligned} \tag{3.6}$$

where we use the definition (1.10) of $G(\Pi_x)$, and

$$\Pi_x := W^{-1}x^{-D}\Re(AA^*)x^{-D^*}W^{-1} = x^{-M}x^{-M^*} \tag{3.7}$$

with

$$M = W^{-1}DW. \tag{3.8}$$

**Remark 3.1** A simpler way to write (3.5) and (3.6) would be to replace $W = S_R\Lambda_R S_R^*$ by $S_R\Lambda_R$. Note that, with our choice, $W$ is positive definite. The relation (3.6) then takes the form (1.8).

The relation (3.6) describes the first set $G_{H,1}$ in the intersection (3.1). Instead of describing the second set $G_{H,2}$ separately, it is more convenient to think of the latter as imposing additional conditions on the elements of $G_{H,1}$. In this regard, observe first that, for any $y > 0$,

$$G_{H,1} = y^D G_{H,1} y^{-D}, \tag{3.9}$$

which simply follows by observing that the condition

$$x^{-D}\Re(AA^*)x^{-D^*} = Cx^{-D}\Re(AA^*)x^{-D^*}C^*,\ x > 0,$$

defining the set $G_{H,1}$, is equivalent to

$$x^{-D}\Re(AA^*)x^{-D^*} = (y^D C y^{-D})x^{-D}\Re(AA^*)x^{-D^*}(y^{-D^*}C^* y^{D^*}),\ x > 0.$$

Using the relation (3.9), all $C \in G_{H,1}$ satisfy the relation

$$x^{-D}\Im(AA^*)x^{-D^*} = Cx^{-D}\Im(AA^*)x^{-D^*}C^*,\ x > 0, \tag{3.10}$$

defining the set $G_{H,2}$, if and only if all $C \in G_{H,1}$ satisfy the same relation with $x = 1$. Considering the form (3.5) of $C \in G_{H,1}$, this imposes additional conditions on the orthogonal matrices $O$. Substituting (3.5) into the relation (3.10) with $x = 1$, we obtain that

$$\Im(AA^*) = WOW^{-1}\Im(AA^*)W^{-1}O^*W,$$

i.e.,

$$OW^{-1}\Im(AA^*)W^{-1} = W^{-1}\Im(AA^*)W^{-1}O,$$

or

$$O \in G(\Pi_I), \tag{3.11}$$

where

$$\Pi_I = W^{-1}\Im(AA^*)W^{-1}. \tag{3.12}$$

By the expressions (3.1), (3.6) and the discussion above, we arrive at the following general result on the structure of symmetry groups of OFBMs, and in particular, on the form of the conjugacy matrix $W$.

**Theorem 3.1** *Consider an OFBM given by the spectral representation (1.3), and suppose that the matrix $A$ satisfies the assumption (2.4). Then, its symmetry group $G_H$ can be expressed as*

$$G_H = W\Big(\bigcap_{x>0} G(\Pi_x) \cap G(\Pi_I)\Big)W^{-1}, \tag{3.13}$$

*where $W$ is defined in (2.3), and $\Pi_x$ and $\Pi_I$ are given in (3.7) and (3.12), respectively.*

The intersection over uncountably many $x > 0$ in (3.13) can be replaced by a countable intersection in a standard way. We have $O \in \cap_{x>0} G(\Pi_x)$ if and only if

$$Ox^{-M}x^{-M^*} = x^{-M}x^{-M^*}O, \; x > 0. \tag{3.14}$$

Writing $x^{-M} = \sum_{k=0}^{\infty} M^k(-\ln x)^k/k!$, the relation (3.14) is equivalent to

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} OM^{k_1}(M^*)^{k_2}\frac{(-\ln x)^{k_1}}{k_1!}\frac{(-\ln x)^{k_2}}{k_2!} = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} M^{k_1}(M^*)^{k_2}O\frac{(-\ln x)^{k_1}}{k_1!}\frac{(-\ln x)^{k_2}}{k_2!}$$

or, with $k_1 = k$, $k_1 + k_2 = m$,

$$\sum_{m=0}^{\infty} O\sum_{k=0}^{m} M^k(M^*)^{m-k}\frac{1}{k!(m-k)!}(-\ln x)^m = \sum_{m=0}^{\infty}\sum_{k=0}^{m} M^k(M^*)^{m-k}O\frac{1}{k!(m-k)!}(-\ln x)^m.$$

Equivalently,

$$O\Pi^{(m)} = \Pi^{(m)}O, \; m \geq 1, \tag{3.15}$$

where

$$\Pi^{(m)} = \sum_{k=0}^{m}\binom{m}{k}M^k(M^*)^{m-k}. \tag{3.16}$$

Theorem 3.1 can now be reformulated as follows.

**Theorem 3.2** *Consider an OFBM given by the spectral representation (1.3), and suppose that the matrix $A$ satisfies the assumption (2.4). Then, its symmetry group $G_H$ can be expressed as*

$$G_H = W\Big(\bigcap_{m=1}^{\infty} G(\Pi^{(m)}) \cap G(\Pi_I)\Big)W^{-1}, \tag{3.17}$$

*where $W$ is defined in (2.3), and $\Pi^{(m)}$ and $\Pi_I$ are given in (3.16) and (3.12), respectively.*

**Remark 3.2** Note that the matrix $\Pi_x$ in (3.7) is positive definite. On the other hand, the matrix $\Pi^{(m)}$ in (3.16) is symmetric because so are the terms

$$\binom{m}{k}M^k(M^*)^{m-k} + \binom{m}{m-k}M^{m-k}(M^*)^k$$

defining $\Pi^{(m)}$. However, $\Pi^{(m)}$ is not positive definite in general. For example, with $\Re(AA^*) = I$ and normal $D$, we have

$$\Pi^{(m)} = (D + D^*)^m, \tag{3.18}$$

which is not positive definite (not even for $m = 1$). Note also that $\Pi_I$ is skew-symmetric, hence normal and diagonizable.

# 4 On maximal and minimal symmetry groups

An operator self-similar process $X$ is said to be of maximal type, or elliptically symmetric, if its symmetry group $G_X$ is conjugate to $O(n)$. At the other extreme, a zero mean (Gaussian) o.s.s. process is said to be of minimal type if its symmetry group is $\{I, -I\}$. We shall examine here these symmetry structures in the case of OFBMs. First, we characterize maximal symmetry in terms of the spectral parametrization of OFBMs. Second, we analyze minimal symmetry OFBMs through a topological lens.

## 4.1 OFBMs of maximal type

The following theorem is the main result of this subsection. Recall the definition of single parameter OFBMs in Example 2.1.

**Theorem 4.1** *Consider an OFBM given by the spectral representation (1.3), and suppose that the matrix $A$ satisfies the assumption (2.4). If an OFBM is of maximal type, then it is a single parameter OFBM up to a conjugacy by a positive definite matrix. Moreover, this happens if and only if*

$$\Im(AA^*) = 0, \quad -(D - dI)\Re(AA^*) = \Re(AA^*)(D^* - dI), \tag{4.1}$$

*for some real $d$.*

**Remark 4.1** Conversely, an OFBM which is a single parameter OFBM (up to a positive definite conjugacy) is of maximal type (see Example 2.1). We also point out that we have a proof of the first claim in Theorem 4.1 which does not make use of spectral representations and dispenses with the assumption (2.4). In the proof of Theorem 4.1 we use spectral representations in order to illustrate how the main results of Section 3 can be used.

PROOF: In view of Proposition A.1 and (3.13), maximal type occurs if and only if

$$\Pi_x = \lambda_x I, \ x > 0, \quad \Pi_I = \lambda I, \quad \lambda_x, \lambda \in \mathbb{R}. \tag{4.2}$$

Note that

$$\Pi_I = \lambda I \quad \Leftrightarrow \quad \Im(AA^*) = \lambda \Re(AA^*) \quad \Leftrightarrow \quad \lambda = 0 \quad \Leftrightarrow \quad \Im(AA^*) = 0. \tag{4.3}$$

Moreover,

$$\Pi_x = \lambda_x I, \ x > 0 \quad \Leftrightarrow \quad \lambda_x \Re(AA^*) = x^{-D}\Re(AA^*)x^{-D^*}, \tag{4.4}$$

which implies that, for any $x_1, x_2 > 0$,

$$\lambda_{x_1x_2}\Re(AA^*) = (x_1x_2)^{-D}\Re(AA^*)(x_1x_2)^{-D^*} = x_2^{-D}\lambda_{x_1}\Re(AA^*)x_2^{-D^*} = \lambda_{x_1}\lambda_{x_2}\Re(AA^*).$$

Hence, under assumption (2.4), $\lambda_{x_1x_2} = \lambda_{x_1}\lambda_{x_2}$, $x_1, x_2 > 0$. Moreover, the function $\log(\lambda_{\exp(\cdot)})$ is additive over $\mathbb{R}$, and it is measurable (since it is continuous). As a consequence, by Theorem 1.1.8 in Bingham et al. (1987), p. 5, there exists a real $d$ such that $\log(\lambda_{\exp(\cdot)}) = -2d(\cdot)$, i.e., $\lambda_x = x^{-2d}$. In particular,

$$x^{-D}\Re(AA^*)x^{-D^*} = x^{-2d}\Re(AA^*). \tag{4.5}$$

Relations (4.3) and (4.5) imply that the covariance structure of OFBM can be written as

$$EB_H(t)B_H(s)^* = \int_{\mathbb{R}} \frac{e^{itx}-1}{ix}\frac{e^{-isx}-1}{-ix}|x|^{-2d}\Re(AA^*)dx.$$

In view of $\Re(AA^*) = W^2$ and (2.7), this shows that $B_H$ is a single parameter OFBM up to a conjugacy.

Finally, note from above that $\Pi_x = \lambda_x I$, $x > 0$, is equivalent to $x^{-D}\Re(AA^*)x^{-D^*} = x^{-2d}\Re(AA^*)$ or $x^{D-dI}\Re(AA^*) = \Re(AA^*)x^{-(D^*-dI)}$ for $x > 0$ and some real $d$. The latter is equivalent to $(D - dI)\Re(AA^*) = -\Re(AA^*)(D^* - dI)$ for some real $d$. □

**Remark 4.2** Theorem 6 in Hudson and Mason (1982) shows that every maximal symmetry o.s.s. process has an exponent of the form $hI$, $h \in \mathbb{R}$. For the case of OFBMs, the proof of Theorem 4.1 retrieves this result (see expression (4.5)). Moreover, it is clear that, for a maximal symmetry OFBM $B_H$ (or, as a matter of fact, for any maximal symmetry o.s.s. process), for any $H \in \mathcal{E}(B_H)$, $W^{-1}HW$ is normal, since $W^{-1}(H - hI)W \in so(n)$ (see also Section 5 for further results on the structure of exponents for dimensions $n = 2$ and $n = 3$).

## 4.2 OFBMs of minimal type: the topologically general case

In view of Theorem 3.1, an OFBM is of minimal type if and only if

$$\bigcap_{x>0} G(\Pi_x) \cap G(\Pi_I) = \{I, -I\},$$

and, in particular, if

$$\bigcap_{x>0} G(\Pi_x) = \{I, -I\}. \tag{4.6}$$

We shall focus here on the relation (4.6) with the following related goals in mind.

The first goal is to provide (practical) conditions for (4.6) to hold and, hence, for an OFBM to be of minimal type. This is a non-trivial problem. The structure of $G(\Pi_x)$ depends on both the eigenvalues and the eigenspaces of $\Pi_x$, which are arbitrary in principle. Moreover, their explicit calculation becomes increasingly difficult with dimension. To shed light on (4.6), we take up an idea from Lie group theory: a lot of information about $M$ in the expression $\Pi_x = x^{-M}x^{-M^*}$ (see (3.7)) is available at the vicinity of the identity in the Lie group, i.e., as $x \to 1$. The general approach we take is to study the behavior of the logarithm of $\Pi_x$ through the Baker-Campbell-Hausdorff formula, valid in a vicinity of the origin of the associated Lie algebra. The characterization of the behavior of the eigenspaces of $\Pi_x$ will then be retrieved by turning back to the Lie group through the exponential map.

Initially, our conditions for the relation (4.6) to hold are in terms of the matrix $M$, and not directly in terms of $H$ and $A$. Our second goal in this section is to show that these conditions on $M$ yield "most" OFBMs in terms of the parametrization $M$, and then relate them back to $H$ and $A$. The term "most" is in the topological sense, i.e., except on a meager set. This result should not be surprising: if $\cap_{x>0}G(\Pi_x)$ has non-trivial structure, then this imposes extra conditions on $M$ (or $D$, $W$) as in Section 4.1. Though not surprising, formalizing this fact is not straightforward, as shown here. This second goal leads to the main result of this section, which, for the sake of clarity, we now briefly describe. In analogy with the assumption (1.2), consider the set

$$\mathcal{D} = \Big\{D \in M(n,\mathbb{R}) : -\frac{1}{2} < \Re(d_k) < \frac{1}{2}, k = 1,\dots,n\Big\}, \tag{4.7}$$

where $d_1,\dots,d_n$ denote the charateristic roots of $D$. Theorem 4.2 below states the existence of a set $\mathcal{M} \subseteq M(n,\mathbb{R})$ such that, for all $D$ and positive definite $W$ such that $W^{-1}DW \in \mathcal{M} \cap \mathcal{D}$, the OFBM with spectral parametrization $D$ and $\Re(AA^*) := W^2$ has minimal symmetry. Moreover, $\mathcal{M} \cap \mathcal{D}$ is an open set (of parameters), and it is dense in $\mathcal{D}$. Therefore, $\mathcal{M}^c \cap \mathcal{D}$ is a meager set. Conversely, every $M \in \mathcal{M} \cap \mathcal{D}$ gives a minimal symmetry OFBM through an appropriate spectral parametrization.

The rest of this section is dedicated to developing these ideas, as well as the framework behind them. Hereinafter, unless otherwise stated, we impose no restrictions on the eigenvalues of $M$, i.e., the expression $\Pi_x = x^{-M}x^{-M^*}$ is taken for any $M \in M(n,\mathbb{R})$. Consider the decomposition of the latter space into the direct sum

$$M(n,\mathbb{R}) = \mathcal{S}(n,\mathbb{R}) \oplus so(n), \tag{4.8}$$

where $\mathcal{S}(n,\mathbb{R})$ is the space of symmetric matrices. For $M \in M(n,\mathbb{R})$, denote

$$M = S + L, \tag{4.9}$$

where $S = (M + M^*)/2$, $L = (M - M^*)/2$ are, respectively, the symmetric and skew-symmetric parts of $M$. Let also

$$\mathcal{S}_{\neq} := \{S \in \mathcal{S}(n, \mathbb{R}) : S \text{ has pairwise different eigenvalues}\}, \tag{4.10}$$

$$\mathcal{L}_{\neq} := \{L \in so(n) : L \text{ has pairwise different eigenvalues}\}. \tag{4.11}$$

The next proposition shows that for an appropriately chosen $M$, the centralizer of the family $\Pi_x$ is minimal. In the proof, the symbol $[\cdot, \cdot]$ denotes the commutator. Since the point $x = 1$ is a singularity in the sense that all the information about $M$ from $\Pi_x = x^{-M} x^{-M^*}$ is lost at it, the idea is to analyze the behavior of $\Pi_x$ for $x$ in a close vicinity of 1.

**Proposition 4.1** *Let $M = S + L \in M(n, \mathbb{R})$ as in (4.9) and suppose that $S \in \mathcal{S}_{\neq}$. Assume also that $S$ and $[L, S]$ do not share eigenvectors. Then, the relation (4.6) holds, that is,*

$$\bigcap_{x>0} G(\Pi_x) = \{I, -I\}.$$

PROOF: Note that $M + M^* = 2S$, and that

$$[M, M^*] = MM^* - M^*M = (S + L)(S - L) - (S - L)(S + L) = 2[L, S].$$

Since the mapping $M \mapsto \exp(M)$ is a $C^\infty$ homeomorphism of some neighborhood of 0 in the Lie algebra of $GL(n, \mathbb{R})$ onto some neighborhood $U$ of the identity $I$ in $GL(n, \mathbb{R})$, then its inverse function Log is well-defined on $U$. Therefore, by the Baker-Campbell-Hausdorff formula, for small enough $\log(x)$ we have

$$\mathrm{Log}(\exp(-\log(x)M)\exp(-\log(x)M^*)) = -\log(x)(M + M^*) + \frac{1}{2}[-\log(x)M, -\log(x)M^*]$$

$$+O(\log^3(x)) = -\log(x)(M + M^*) + \log^2(x)\frac{1}{2}[M, M^*] + O(\log^3(x))$$

(see Hausner and Schwartz (1968), p. 63 and pp. 68-69). We claim that, for some $\delta > 0$, the family

$$(M + M^*) - \frac{1}{2}\log(x)[M, M^*] + O(\log^2(x)), \quad x \in B(1, \delta)\backslash\{1\}, \tag{4.12}$$

does not share eigenvectors with $M + M^*$. In fact, assume by contradiction that, for a sequence $x_k \to 1$, there exists a sequence $\{v_k\}$ of unit norm vectors which are eigenvectors of (4.12), associated with the eigenvalues $\lambda_k$, and of $M + M^*$, associated with the eigenvalues $\lambda_k^S$. By passing to a subsequence if necessary, we can assume $\lambda_k^S = \lambda^S$. Then,

$$(M + M^*)v_k - \frac{1}{2}\log(x_k)[M, M^*]v_k + O(\log^2(x_k))v_k = \lambda_k v_k.$$

Therefore,

$$-\frac{1}{2}\log(x_k)[M, M^*]v_k + O(\log^2(x_k))v_k = (\lambda_k - \lambda^S)v_k,$$

or

$$-\frac{1}{2}[M, M^*]v_k + O(\log(x_k))v_k = \frac{\lambda_k - \lambda^S}{\log(x_k)}v_k. \tag{4.13}$$

Since $\{v_k\}$ is a sequence in $S^{n-1}$, we can pass to a subsequence if necessary to obtain $v_k \to v_0 \in S^{n-1}$.

Therefore, regarding the left-hand side of expression (4.13), we have

$$-\frac{1}{2}[M, M^*]v_k + O(\log(x_k))v_k \to -\frac{1}{2}[M, M^*]v_0.$$

As a consequence, regarding the right-hand side of expression (4.13), $\frac{\lambda_k - \lambda^S}{\log(x_k)}$ must have a limit, which we denote by $\lambda$. Therefore, $v_0$ is an eigenvector of $[M, M^*]$ associated with the eigenvalue $\lambda$. Now note that $\{v_k\}$ is a sequence of eigenvectors of $M + M^*$, all of them associated with the eigenvalue $\lambda^S$. Therefore, $v_0$ is also an eigenvector of $M + M^*$, which implies that $S$ and $[L, S]$ share an eigenvector (contradiction). Therefore,

$$\mathrm{Log}(\exp(-\log(x)M)\exp(-\log(x)M^*))$$

does not share eigenvectors with $-\log(x)(M + M^*)$, $x \in B(1, \delta)\backslash\{1\}$.

By taking another $\delta' < \delta$ if necessary, we know that the eigenvalues of the matrix $(M+M^*) - \frac{1}{2}\log(x)[M, M^*] + O(\log^2(x))$, $x \in B(\delta', 1)\backslash\{1\}$, are pairwise different, since they are approaching those of $M + M^*$ (Lemma B.1). Thus, we may take $x_0 \in B(1, \delta')\backslash\{1\}$ such that the eigenvalues of $\Pi_{x_0}$ are pairwise different.

From the development above, the matrices $(M + M^*) - \frac{1}{2}\log(x_0)[M, M^*] + O(\log^2(x_0))$ and $M + M^*$ have no eigenvectors in common. However, since by Lemma B.2 the eigenvectors of $(M + M^*) - \frac{1}{2}\log(x)[M, M^*] + O(\log^2(x))$ converge to those of $M + M^*$ as $x \to 1$ (in the sense specified in Lemma B.2), there must exist $x_1 \in B(1, \delta')\backslash\{1\}$, $x_1 \neq x_0$, such that $\mathrm{Log}(\Pi_{x_0})$ and $\mathrm{Log}(\Pi_{x_1})$ also share no eigenvectors, and $\mathrm{Log}(\Pi_{x_1})$ also has pairwise different eigenvalues.

Since $\exp(\mathrm{Log}(\Pi_x)) = \Pi_x$ (for $x$ close enough to 1), then it is also true that $\Pi_{x_0}$, $\Pi_{x_1}$ each has pairwise different eigenvalues and that they share no eigenvectors. Thus, for $O \in O(n)$ to commute with both $\Pi_{x_0}$ and $\Pi_{x_1}$, it must be a multiple of the identity, by Lemma 2.1. $\square$

The following technical lemmas will be used in the sequel.

**Lemma 4.1** *Let $L \in \mathcal{L}_{\neq}$. If $n$ is even, then $L$ has no real eigenvectors.*

Proof: Let

$$U_2 = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \in U(2). \tag{4.14}$$

Any $L \in \mathcal{L}_{\neq} \subseteq so(n)$ can be written in the form $L = OU\mathrm{diag}(ia_1, -ia_1, \ldots, ia_{n/2}, -ia_{n/2})U^*O^*$, for some $O \in O(n)$, where $a_k \in \mathbb{R}, k = 1, \ldots, n/2$, and $U = \mathrm{diag}(U_2, \ldots, U_2) \in U(n)$, where $n/2$ blocks $U_2$ are used. Denote by $o_j$ the $j$th column of $O$. Since the eigenvalues of $L$ are pairwise different, the eigenvectors of $L$ necessarily have the form

$$z\Big(o_j\frac{\sqrt{2}}{2} \pm io_{j+1}\frac{\sqrt{2}}{2}\Big), \quad z \in \mathbb{C}\backslash\{0\}, \quad j = 1, 3, \ldots, \frac{n}{2} - 1.$$

However, $z(o_j\frac{\sqrt{2}}{2} \pm io_{j+1}\frac{\sqrt{2}}{2}) \in \mathbb{R}^n$ would imply that

$$\Re(z)o_{j+1} = \mp\Im(z)o_j,$$

i.e., either $z = 0$ or $o_j$, $o_{j+1}$ are linearly dependent, both of which lead to a contradiction. $\square$

**Lemma 4.2** *Consider $S \in \mathcal{S}_{\neq}$ and $L \in so(n)$. If $S$ and $[L, S]$ share a (real) eigenvector $v$, then $v$ is an eigenvector of $L$.*

PROOF: Without loss of generality, assume $v \in S^{n-1}$, and denote by $\lambda_S$ the eigenvalue of $S$ with which $v$ is associated. Since $v$ is also an eigenvector of $[L, S]$, there exists $\lambda$ such that

$$(LS - SL)v = \lambda v,$$

i.e.,

$$\lambda_S Lv - SLv = \lambda v.$$

Since $S \in \mathcal{S}(n, \mathbb{R})$, then

$$v^* S = (Sv)^* = (\lambda_S v)^* = v^* \lambda_S.$$

Thus,

$$0 = \lambda_S v^* Lv - v^* \lambda_S Lv = \lambda.$$

Therefore,

$$(\lambda_S I - S)Lv = 0 \in \mathbb{R}^n.$$

Note that

$$(\lambda_S I)(Lv) = S(Lv).$$

Since the eigenvalues of $S$ are pairwise different, then $Lv$ is in the eigenspace generated by $v$. Thus, $L$ acts on $v$ by multiplying it by a real scalar, i.e., $v$ is also a (real) eigenvector of $L$. □

The following result is an immediate consequence of Lemmas 4.1 and 4.2.

**Corollary 4.1** *Let $S \in \mathcal{S}_{\neq}$, $L \in \mathcal{L}_{\neq}$. If $n$ is even, then $S$ and $[L, S]$ share no eigenvectors.*

The next result introduces a set (expression (4.15) below) that will be important in the subsequent development.

**Lemma 4.3** *Let $n$ be odd. Let $S \in \mathcal{S}_{\neq}$, and denote by $o_1, \ldots, o_n$ a collection of $n$ orthonormal eigenvectors of $S$. Then, the set of $L \in \mathcal{L}_{\neq}$ such that $S$ and $L$ share a (real) eigenvector $v$ has the form*

$$\bigcup_{j=1}^{n} \mathcal{L}(o_j), \tag{4.15}$$

*where*

$$\mathcal{L}(o_j) = \{L \in \mathcal{L}_{\neq} : o_j \text{ is an eigenvector of } L \text{ associated with the eigenvalue } 0\}. \tag{4.16}$$

PROOF: For every $L \in \mathcal{L}_{\neq}$, there exist $O \in O(n)$ and $a_j \in \mathbb{R}\backslash\{0\}$, $j = 1, ..., (n-1)/2$, such that $L = OU\mathrm{diag}(ia_1, -ia_1, \ldots, ia_{(n-1)/2}, -ia_{(n-1)/2}, 0)U^*O^*$, where $U = \mathrm{diag}(U_2, \ldots, U_2, 1) \in U(n)$ and $U_2$ is as in expression (4.14). Since the eigenvectors induced by the first $n-1$ columns of $OU$ necessarily have non-trivial imaginary parts, then for $S$ and $L$ to share an eigenvector, it must be the one in the space induced by the last column of $OU$, which is associated with the eigenvalue 0. □

**Example 4.1** For the sake of concreteness, we now briefly describe the form of a set $\mathcal{L}(\cdot)$ in the lowest-dimensional non-trivial setting, i.e., $n = 3$. Assume $S$ is diagonal, and that the canonical vector $e_3 = (0, 0, 1)'$ is an eigenvector $S$ and $L$ share. Then, since

$$L = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix},$$

the equality $Le_3 = 0$ implies that $b = c = 0$, which is the general form of the matrices in the subspace $\mathcal{L}(e_3) \cup \{0\}$ of $so(3)$. It is clear that an analogous argument applies when one picks either $e_1$ or $e_2$ as the common eigenvector, and the general case treated in Lemma 4.3 follows by considering a suitable change of coordinates $O \in O(3)$.

Proposition 4.2 below establishes the topological properties of the class of "well-behaved" exponents $M$, i.e., those that will eventually be associated with minimal symmetry OFBMs. Its proof is based on the next two lemmas.

**Lemma 4.4** *(i) The set $\mathcal{S}_{\neq}$ is an open, dense set in (the relative topology of) $\mathcal{S}(n, \mathbb{R})$.*

*(ii) Let $n$ be odd. For any orthonormal vectors $o_1, \ldots, o_n$ in $\mathbb{R}^n$, the set*

$$\Big(\bigcup_{j=1}^{n} \mathcal{L}(o_j)\Big)^c \cap \mathcal{L}_{\neq},$$

*is an open, dense set in (the relative topology of) $so(n)$.*

Proof: We only prove $(ii)$, since $(i)$ can be handled with a similar, but simpler, argument.

First, we show openness. Let $L_0 \in (\cup_{j=1}^{n} \mathcal{L}(o_j))^c \cap \mathcal{L}_{\neq}$. Assume by contradiction that there exists a sequence $L_k \in (\cup_{j=1}^{n} \mathcal{L}(o_j)) \cup (\mathcal{L}_{\neq})^c$ such that $L_k \to L_0$. Since by Lemma B.1, the eigenvalues of $L_k$ converge to those of $L_0$, then without loss of generality we can assume that the latter are pairwise distinct, i.e., $L_k \in \cup_{j=1}^{n} \mathcal{L}(o_j)$. By passing to a subsequence if necessary, we can assume that, for all $k$, $L_k \in \mathcal{L}(o_j)$ for some fixed $j$, i.e., $L_k o_j = 0$. Since, by assumption, $L_k \to L_0$, then $L_0 o_j = 0$ (contradiction).

We now show denseness. Take $L \in so(n)$, and fix the orthonormal vectors $o_1, \ldots, o_n \in \mathbb{R}^n$. Consider the case when $L \in \cup_{j=1}^{n} \mathcal{L}(o_j)$. Without loss of generality, we can make the further assumption that $L \in \mathcal{L}(o_n)$. Then, we can write $L = OU\mathrm{diag}(ia_1, -ia_1, \ldots, ia_{(n-1)/2}, -ia_{(n-1)/2}, 0)U^* O^*$, where $o_n$ is the last column of the matrix $O$, and $U = \mathrm{diag}(U_2, \ldots, U_2, 1)$, with $U_2$ being as in (4.14). Consider a sequence $O_k \in O(n)$ whose columns are all not in the subspace generated by any column of $O$, and, additionally, such that $O_k \to O$. Then,

$$L_k = O_k U \mathrm{diag}(ia_1, -ia_1, \ldots, ia_{(n-1)/2}, -ia_{(n-1)/2}, 0)U^* O_k^* \to L,$$

and $L_k \in (\cup_{j=1}^{n} \mathcal{L}(o_j))^c \cap \mathcal{L}_{\neq}$.

For the case when $L$ has repeated eigenvalues, we may use the argument above but choosing an appropriate sequence of eigenvalues for $L_k$ that converge to the eigenvalues of $L$. □

We now define a correspondence (set-valued function) that maps the set $\mathcal{S}_{\neq}$ to the set of skew-symmetric matrices used in Lemma 4.4 above.

**Definition 4.1** Let $\mathcal{P}$ denote the class of all subsets of a set. Define the correspondence (set-valued function)

$$l : \mathcal{S}_{\neq} \to \mathcal{P}(so(n)),$$

$$S \mapsto l(S) = \begin{cases} (\bigcup_{j=1}^{n} \mathcal{L}(o_j))^c \cap \mathcal{L}_{\neq}, & n \text{ is odd}, \\ so(n), & n \text{ is even}, \end{cases}$$

where $o_1, \ldots, o_n$ represent orthonormal eigenvectors of $S$.

Regarding the case when $n$ is odd, the correspondence $l$ is well-defined because, for a given $S \in S_{\neq}$, any orthonormal eigenvectors $o_1, \ldots, o_n$ of $S$ give the same image $l(S)$.

The following lemma sheds light on the graph of the correspondence $l(\cdot)$ from a topological standpoint.

**Lemma 4.5** *Let $l(\cdot)$ be the correspondence in Definition 4.1. Then,* $\mathrm{Graph}(l) := \{(S, L) : S \in \mathcal{S}_{\neq}, L \in l(S)\}$ *is open and dense in* $(\mathcal{S}(n, \mathbb{R}), so(n))$.

PROOF: Openness is a consequence of the fact that, if $S_0 \in \mathcal{S}_{\neq}$, then, as $S_k \to S_0$, the eigenvalues of $S_k$ converge to those of $S_0$ (in the sense of Lemma B.1). Indeed, assume by contradiction that there exists $(S_0, L_0) \in \mathrm{Graph}(l)$ such that, for some sequence $(S_k, L_k) \notin \mathrm{Graph}(l)$,

$$(S_k, L_k) \to (S_0, L_0).$$

Note that there cannot be a subsequence $\{S_{k'}\} \subseteq \mathcal{S}^c_{\neq}$ such that $S_{k'} \to S_0$ (since this contradicts the openness of $\mathcal{S}_{\neq}$ established in Lemma 4.4). Thus, if $n$ is even, this is a contradiction with the fact that $(S_k, L_k) \notin \mathrm{Graph}(l)$, and the proof ends here. On the other hand, if $n$ is odd, this implies that we must have $L_k \notin l(S_k)$, $k \in \mathbb{N}$. Since $L_0 \in \mathcal{L}_{\neq}$, then by Lemma B.1 we can assume without loss of generality that $L_k \in \mathcal{L}_{\neq}$. Thus, there exists $j_k \in \{1, \ldots, n\}$ such that

$$L_k(o^k_{j_k}) = 0,$$

i.e., some eigenvector $o^k_{j_k} \in S^{n-1}$ of $S_k$ is in the kernel of $L_k$. By Lemma B.1, we can assume that the associated sequence of eigenvalues $\{\lambda_{j_k}\}$ of $S_k$ converges to some eigenvalue $\lambda_1$ of $S_0$. Moreover, since $o^k_{j_k} \in S^{n-1}$, by passing to a subsequence if necessary, we have that $o^k_{j_k} \to o_1$, where $o_1$ is some vector in $S^{n-1}$. Then,

$$S_k o^k_{j_k} = \lambda_{j_k} o^k_{j_k},$$

where $S_k o^k_{j_k} \to S_0 o_1$ and $\lambda_{j_k} o^k_{j_k} \to \lambda_1 o_1$. Consequently, $o_1$ is an eigenvector of $S_0$. Moreover,

$$0 = L_k(o^k_{j_k}) \to L_0(o_1).$$

Therefore, $L_0 \in (l(S_0))^c$ (contradiction).

Denseness comes immediately from Lemma 4.4. $\square$

The next proposition puts $\mathrm{Graph}(l)$ back into the original space $M(n, \mathbb{R})$ in the form of a direct sum, rephrases the topological statement of Lemma 4.5, and connects the latter to the problem of proving (4.6).

**Proposition 4.2** *Let*

$$\mathcal{M} = \{M \in M(n, \mathbb{R}) : M \in \mathcal{S}_{\neq} \oplus l(\mathcal{S}_{\neq})\}. \tag{4.17}$$

*Then,*

*(i) $\mathcal{M}$ is an open, dense subset of $M(n, \mathbb{R})$. Consequently, $\mathcal{M}^c$ is a meager set and $\mathcal{M}$ is a $n^2$-dimensional $C^\infty$ manifold in $\mathbb{R}^{n^2} \cong M(n, \mathbb{R})$.*

*(ii) relation (4.6) holds for all $M \in \mathcal{M}$.*

Proof: We first show part $(i)$. Define the norm $\|M\|_{\oplus} = \|S\| + \|L\|$, where $\|\cdot\|$ is the spectral matrix norm. Expression (4.8) implies that $\|\cdot\|_{\oplus}$ is well-defined. By the equivalence of matrix norms, it suffices to show $(i)$ with respect to $\|\cdot\|_{\oplus}$. Assume by contradiction that there is some $M_0 = S_0 + L_0 \in \mathcal{M}$ and a sequence $\{M_k\} \subseteq \mathcal{M}^c$ such that $\|M_k - M_0\|_{\oplus} \to 0$. However, such convergence holds if and only if $\|S_k - S_0\| \to 0$ and $\|L_k - L_0\| \to 0$. Since, for each $k$, either $S_k \notin \mathcal{S}_{\neq}$ or $L_k \notin l(S_k)$, then this contradicts the openness of Graph$(l)$ (Lemma 4.5). Denseness can be addressed in a similar fashion, and the geometric statement is immediate.

Part $(ii)$ is a consequence of Proposition 4.1, Corollary 4.1 and Lemma 4.3. $\square$

**Example 4.2** To construct an example of $M \in \mathcal{M}$, we turn again to the case when $n = 3$ (see Example 4.1). Take $S = \text{diag}(s_1, s_2, s_3)$, where the (real) eigenvalues are pairwise different. Then, take any $L \in so(3)\backslash\{0\}$ *not* having one of the Euclidean canonical vectors $e_1$, $e_2$, $e_3$ in its unidimensional kernel. In other words, we can*not* take a matrix $L$ of one of the forms

$$\begin{pmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ -b & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -c & 0 \end{pmatrix},$$

where $a, b, c \neq 0$. Now set $M = S + L$.

In order to make the claim about the general minimality of the symmetry groups of OFBMs, we need to restrict the parameter space, as in (1.2). For this purpose, we consider the set $\mathcal{D}$ in (4.7). The following is the main result of this section. It shows that, except possibly when the parametrization is taken on a meager set, OFBMs are of minimal symmetry.

**Theorem 4.2** *For all $D, W \in M(n, \mathbb{R})$, where $W$ is positive definite, and such that*

$$W^{-1}DW \in \mathcal{M} \cap \mathcal{D},$$

*the associated OFBM with spectral parametrization $D$ and $\Re(AA^*) := W^2$ has minimal symmetry. The set $\mathcal{M}\cap\mathcal{D}$ is open, and, in particular, it is an $n^2$-dimensional $C^\infty$ manifold in $\mathbb{R}^{n^2} \cong M(n, \mathbb{R})$. Moreover, it is also a dense subset of $\mathcal{D}$ (in the relative topology of $\mathcal{D}$). As a consequence, $\mathcal{M}^c \cap \mathcal{D}$ is a meager set.*

*Conversely, every $M \in \mathcal{M} \cap \mathcal{D}$ gives rise to a minimal symmetry OFBM through the spectral parametrization $D := M$, $W := I$.*

Proof: By the convergence of eigenvalues ensured by Lemma B.1, $\mathcal{D}$ is an open set. Therefore, by Proposition 4.2, $\mathcal{M} \cap \mathcal{D}$ is also an open set. The geometric statement is straightforward. Furthermore, since $\mathcal{M}$ is dense in $M(n, \mathbb{R})$, then $\mathcal{M} \cap \mathcal{D}$ must also be dense in the relative topology of the open set $\mathcal{D}$.

The converse is an immediate consequence of Proposition 4.2. $\square$

# 5 Classification in dimensions $n = 2$ and $n = 3$

Theorems 3.1 and 3.2 describe the general structure of symmetry groups of OFBMs. The cases of maximal and minimal symmetry groups were studied in Section 4. In this section, we are interested in identifying all the possible "intermediate" symmetry groups. We shall describe their structure in dimensions $n = 2$ and $n = 3$, and make some comments about higher dimensions.

## 5.1 Dimension $n = 2$

When $n = 2$, the contribution of the term $G(\Pi_I)$ in Theorems 3.1 and 3.2 can be easily described, as the next two results show.

**Lemma 5.1** *If $\Pi_I \neq 0$, then $G(\Pi_I) = SO(2)$.*

PROOF: Since $\Pi_I$ is skew-symmetric, we have

$$\Pi_I = \lambda \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \lambda \neq 0.$$

Thus, $\Pi_I/\lambda$ is a rotation matrix, and thus $G(\Pi_I) = SO(2)$. $\Box$

Theorems 3.1 and 3.2 can now be reformulated as follows.

**Corollary 5.1** *For $n = 2$, under the assumptions and notation of Theorems 3.1 and 3.2, we have*

$$\begin{aligned} G_H &= W \left\{ \begin{array}{ll} \cap_{x>0} G(\Pi_x) \cap SO(2), & \text{if } \Im(AA^*) \neq 0 \\ \cap_{x>0} G(\Pi_x), & \text{if } \Im(AA^*) = 0 \end{array} \right\} W^{-1} &(5.1)\\ &= W \left\{ \begin{array}{ll} \cap_{m\geq 1} G(\Pi^{(m)}) \cap SO(2), & \text{if } \Im(AA^*) \neq 0 \\ \cap_{m\geq 1} G(\Pi^{(m)}), & \text{if } \Im(AA^*) = 0 \end{array} \right\} W^{-1}. &(5.2)\end{aligned}$$

Next, we study the possible structures of the groups $G(\Pi)$ when $\Pi$ is symmetric (and hence potentially positive definite, as the matrix $\Pi_x$ in (5.1)). Let $\pi_1, \pi_2$ be the two real eigenvalues of $\Pi$. Two cases need to be considered:

$$\begin{array}{ll} \text{Case 2.1:} & \pi_1 = \pi_2, \\ \text{Case 2.2:} & \pi_1 \neq \pi_2. \end{array} \tag{5.3}$$

In Case 2.1, $\Pi = \pi_1 I$ and hence

$$G(\Pi) = O(2). \tag{5.4}$$

In Case 2.2, we can write

$$\Pi = S \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix} S^* = (p_1 \; p_2) \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix} \begin{pmatrix} p_1' \\ p_2' \end{pmatrix},$$

where the columns of the orthogonal matrix $S = (p_1 \; p_2)$ consist of the orthonormal eigenvectors $p_1, p_2$ of $\Pi$. By Theorem 2.1, $B \in O(2)$ commutes with such $\Pi$ if and only if $B = SGS^*$ where $G$ is a diagonal matrix such that $G^2 = I$ ($G^2 = I$ is a consequence of the fact that $B \in O(2)$), or

$$B = S \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} S^* = (p_1 \; p_2) \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} p_1' \\ p_2' \end{pmatrix}.$$

We thus have

$$\begin{aligned} G(\Pi) &= \left\{ I, -I, S \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S^*, S \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} S^* \right\} \\ &= \{I, -I, \mathrm{Ref}(p_1), \mathrm{Ref}(p_2)\}, \end{aligned} \tag{5.5}$$

where $\mathrm{Ref}(p)$ indicates a reflection around the axis spanned by a vector $p$. The expressions (5.4) and (5.5) provide the only possible structures for $G(\Pi)$. Together with Corollary 5.1, this leads to the following result.

**Theorem 5.1** *Consider an OFBM given by the spectral representation (1.3), and suppose that the matrix $A$ satisfies the assumption (2.4). Then, its symmetry group $G_H$ is conjugate to one of the following:*

(2.$a$) *minimal type:* $\{I, -I\}$*;*

(2.$b$) *trivial type:* $\{I, -I, \mathrm{Ref}(p_1), \mathrm{Ref}(p_2)\}$ *for a pair of orthogonal* $p_1, p_2$*;*

(2.$c$) *rotational type:* $SO(2)$*;*

(2.$d$) *maximal type:* $O(2)$*.*

All the types of subgroups described in Theorem 5.1 are non-empty, as we show next. Since OFBMs of maximal and minimal types were studied in general dimension $n$ in Section 4, we now provide examples of OFBMs of only the two remaining types for dimension $n = 2$.

**Example 5.1** (Rotational type) Consider an OFBM with parameters

$$D = dI, \quad \sqrt{2}A_1 \in SO(2)\backslash\{I, -I\}, \quad \sqrt{2}A_2 \in O(2)\backslash SO(2), \tag{5.6}$$

where $d$ is real. Then, we have $\Pi_x = x^{-2d}I$ and $G(\Pi_x) = O(2)$. Since $\Im(AA^*) \neq 0$, Corollary 5.1 yields that $G_H = SO(2)$.

**Example 5.2** (Trivial type) Consider an OFBM with parameters

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \quad A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \tag{5.7}$$

where $d_1 \neq d_2$ are real. Then,

$$AA^* = \begin{pmatrix} |a_1|^2 & 0 \\ 0 & |a_2|^2 \end{pmatrix} = \Re(AA^*), \quad \Im(AA^*) = 0,$$

and

$$\Pi_x = \begin{pmatrix} x^{-2d_1} & 0 \\ 0 & x^{-2d_2} \end{pmatrix},$$

implying that, for $x \neq 1$,

$$G(\Pi_x) = \left\{ I, -I, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Corollary 5.1 then yields

$$G_H = \left\{ I, -I, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right\}. \tag{5.8}$$

Only OFBMs of rotational and maximal types have multiple exponents. Moreover, in view of (1.5), in both cases we have

$$\mathcal{E}(B_H) = H + W so(2) W^{-1}, \tag{5.9}$$

where $H$ is any exponent of the OFBM $B_H$. This relation can be further refined, as the following proposition shows. For this purpose, we need to consider a so-called commuting exponent $H_0 \in \mathcal{E}(B_H)$, i.e., an exponent $H_0$ such that

$$H_0 C = C H_0 \tag{5.10}$$

for all $C \in G_H$. The existence of this useful exponent is ensured by Lemma 2 of Maejima (1998).

**Proposition 5.1** *Consider an OFBM given by the spectral representation (1.3), and suppose that the matrix A satisfies the assumption (2.4). If $\mathcal{E}(B_H)$ is not unique, then the commuting exponents are of the form*

$$H_0 = WU_2\mathrm{diag}(h, \overline{h})U_2^* W^{-1}, \tag{5.11}$$

*where $U_2$ is as in (4.14), and $h \in \mathbb{C}$. In particular,*

$$\mathcal{E}(B_H) = W(U_2\mathrm{diag}(h, \overline{h})U_2^* + so(2))W^{-1}, \tag{5.12}$$

*$H = \Re(h)I \in \mathcal{E}(B_H)$ and $W^{-1}HW$ is normal for any $H \in \mathcal{E}(B_H)$.*

PROOF: If $\mathcal{E}(B_H)$ is not unique, then by Theorem 5.1, $H_0$ commutes with $WSO(2)W^{-1}$. In particular, $H_0$ commutes with $WOW^{-1}$ for $O \in SO(2)\backslash\{I, -I\}$. Since such $O$ is diagonalizable with two complex conjugate eigenvalues, the eigenvectors of $WOW^{-1}$ are also eigenvectors of $H_0$. Thus, $H_0$ can be written as $WU_2\mathrm{diag}(h_1, h_2)U_2^* W^{-1}$. Therefore, since $h_1$, $h_2$ are also the eigenvalues of $U_2\mathrm{diag}(h_1, h_2)U_2^*$, which must only have real entries, a simple calculation shows that $h_1 = \overline{h}_2$, and thus (5.11) holds. This also yields (5.12).

For $H \in \mathcal{E}(B_H)$, (5.12) implies that $W^{-1}HW$ is normal. In particular, we may choose the exponent $H = H_0 + WL_{-\Im(h)}W^{-1} = \Re(h)I$, where $L_s$ is defined in (2.5). □

**Remark 5.1** In the case of OFBMs of rotational type, which have multiple exponents, every exponent is a commuting exponent (compare with Meerschaert and Veeh (1993), p. 721, for the case of operator stable measures).

**Remark 5.2** For general proper Gaussian processes, one can define symmetry sets (groups) in the same way as for o.s.s. processes, and, in particular, show that they are also compact subgroups of $GL(n, \mathbb{R})$. By applying the argument of the proof of Theorem 4.5.3 in Didier (2007), which is based on general commutativity results, instead of spectral filters, one can show that the classification provided by Theorem 5.1 actually holds for the wide class of proper bivariate Gaussian processes.

### 5.2 Dimension $n = 3$

We will make use of the partition of $O(3)$ into the following subsets:

$$SO(3) = \{I\} \cup \mathrm{Rot}_\theta \cup \mathrm{Rot}_\pi, \quad O(3)\backslash SO(3) = \{-I\} \cup \mathrm{Ref}_\theta \cup \mathrm{Ref}_0,$$

where for a vector $p$,

$$\mathrm{Rot}_\theta := \bigcup_{p \in S^{n-1}} \mathrm{Rot}_\theta(p), \quad \mathrm{Rot}_\pi := \bigcup_{p \in S^{n-1}} \mathrm{Rot}_\pi(p),$$

$$\mathrm{Ref}_\theta := \bigcup_{p \in S^{n-1}} \mathrm{Ref}_\theta(p), \quad \mathrm{Ref}_0 := \bigcup_{p \in S^{n-1}} \mathrm{Ref}_0(p),$$

and

$$\begin{aligned}
\mathrm{Rot}_\theta(p) &= \{\text{rotations about the axis } \mathrm{span}_{\mathbb{R}}(p) \text{ by an angle not equal to } \pi\},\\
\mathrm{Rot}_\pi(p) &= \{\text{rotation about the axis } \mathrm{span}_{\mathbb{R}}(p) \text{ by an angle equal to } \pi\},\\
\mathrm{Ref}_\theta(p) &= \{\text{rotations about the axis } \mathrm{span}_{\mathbb{R}}(p) \text{ by an angle not equal to } \pi, \text{ combined with the}\\
&\qquad \text{reflection in the plane through the origin which is perpendicular to the axis}\},\\
\mathrm{Ref}_0(p) &= \{\text{reflection in a plane through the origin, where the plane is perpendicular to } p\}.
\end{aligned} \tag{5.13}$$

From a matrix perspective, for some $p \in S^{n-1}$,

$$\mathrm{Rot}_\theta(p) \cong \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \theta \in (0, 2\pi)\backslash\{\pi\}, \quad \mathrm{Rot}_\pi(p) \cong \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\mathrm{Ref}_\theta(p) \cong \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ \theta \in (0, 2\pi)\backslash\{\pi\}, \quad \mathrm{Ref}_0(p) \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

where $\cong$ indicates conjugacies by orthogonal matrices.

**Remark 5.3** The subscript $\theta$ in $\mathrm{Rot}_\theta$ or $\mathrm{Ref}_\theta$ only indicates that the angle in question is not 0 or $\pi$. Here, $\theta$ does *not* refer to a specific angle. Indeed, even in the case of a fixed $p$, $\mathrm{Rot}_\theta(p)$ and $\mathrm{Ref}_\theta(p)$ are classes of matrices. Also, in the expression $\mathrm{Ref}_0$ we use the subscript 0 to indicate that there is no rotation before reflection through the plane in question.

We first describe the possible structures of $G(\Pi)$ for symmetric matrices $\Pi$ (such as the matrices $\Pi_x$, $x > 0$, in (3.13)). Let $\pi_1, \pi_2, \pi_3$ be the three real eigenvalues of $\Pi$. Three cases need to be considered, namely,

$$\begin{array}{ll} \text{Case 3.1:} & \pi_1 = \pi_2 = \pi_3, \\ \text{Case 3.2:} & \pi_1 = \pi_2 \neq \pi_3, \\ \text{Case 3.3:} & \pi_i \neq \pi_j, \ \ i \neq j, \ \ i, j = 1, 2, 3. \end{array} \tag{5.14}$$

The next proposition gives the form of $G(\Pi)$ in all the above cases.

**Proposition 5.2** *Let $\Pi \in \mathcal{S}(3, \mathbb{R})$. Denote its eigenvectors by $p_i$, $i = 1, 2, 3$, where $S = (p_1\ p_2\ p_3) \in O(3)$. Then,*

*(i) in Case 3.1 in (5.14),*

$$G(\Pi) = O(3); \tag{5.15}$$

*(ii) in Case 3.2 in (5.14),*

$$\begin{aligned} G(\Pi) = \{I, -I\} &\cup (\mathrm{Rot}_\theta(p_3) \cup \mathrm{Ref}_\theta(p_3)) \cup (\mathrm{Rot}_\pi(p_3) \cup \mathrm{Ref}_0(p_3)) \\ &\cup \bigcup_{q \in \mathrm{span}_{\mathbb{R}}\{p_1, p_2\}} (\mathrm{Rot}_\pi(q) \cup \mathrm{Ref}_0(q)); \end{aligned} \tag{5.16}$$

*(iii) in Case 3.3 in (5.14),*

$$G(\Pi) = \{I, -I, \mathrm{Ref}_0(p_1), \mathrm{Ref}_0(p_2), \mathrm{Ref}_0(p_3), \mathrm{Rot}_\pi(p_1), \mathrm{Rot}_\pi(p_2), \mathrm{Rot}_\pi(p_3)\}. \tag{5.17}$$

PROOF: $(i)$ is immediate, so we turn to $(ii)$. In this case, we can write $\Pi = S\mathrm{diag}(\pi_1, \pi_1, \pi_3)S^*$. By Theorem 2.1, $B$ commutes with such $\Pi$ if and only if

$$B = S \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & d \end{pmatrix} S^*, \tag{5.18}$$

where $C = (c_{ij})_{i,j=1,2}$ and $d$ are arbitrary. If we are only interested in orthogonal matrices, this gives $C \in O(2)$, and $d = \pm 1$, which corresponds to the subgroup (5.16). Indeed, the matrices

$\mathrm{Rot}_\theta(p_3)$ and $\mathrm{Rot}_\pi(p_3)$ in (5.16) account for rotations $C \in SO(2)$ and $d = 1$ in (5.18), $\mathrm{Ref}_\theta(p_3)$ and $\mathrm{Ref}_0(p_3)$ in (5.16) account for rotations $C \in SO(2)$ and $d = -1$ in (5.18), and $\mathrm{Rot}_\pi(p_3)$ and $\mathrm{Ref}_0(q)$, $q \in \mathrm{span}_{\mathbb{R}}\{p_1, p_2\}$, account for reflections $C \in O(2)\backslash SO(2)$ and $d = \pm 1$ in (5.18).

Regarding $(iii)$, we can write $\Pi = S\mathrm{diag}(\pi_1, \pi_2, \pi_3)S^*$. By Theorem 2.1, $B \in O(3)$ commutes with such $\Pi$ if and only if

$$B = S \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} S^*.$$

We thus have

$$\begin{aligned} G(\Pi) \;\; = \;\; & \{I, -I, S\mathrm{diag}(-1,1,1)S^*, S\mathrm{diag}(1,-1,1)S^*, S\mathrm{diag}(1,1,-1)S^*, \\ & S\mathrm{diag}(1,-1,-1)S^*, S\mathrm{diag}(-1,1,-1)S^*, S\mathrm{diag}(-1,-1,1)S^*\}, \end{aligned}$$

as stated. $\Box$

The expressions (5.15), (5.16) and (5.17) describe the only possible structures for $G(\Pi)$, all of them, as shown below, being symmetry groups of some non-empty subclass of OFBMs. However, new symmetry groups may arise when one considers intersections of $G(\Pi_x)$ for different values of $x$, and also with $G(\Pi_I)$. In order to provide a full description of symmetry groups of OFBMs in dimension $n = 3$, we first consider the case of time reversible OFBMs, before turning to the general case. As shown in Didier and Pipiras (2010), time reversibility corresponds to the assumption that

$$\Im(AA^*) = 0. \tag{5.19}$$

Under (5.19), the presence of $G(\Pi_I)$ in (3.13) and (3.17) can be ignored.

**Theorem 5.2** *Consider an OFBM given by the spectral representation (1.3), and suppose that the matrix $A$ satisfies the assumptions (2.4) and (5.19). Then, its symmetry group $G_H$ is conjugate by a positive definite matrix $W$ to one of the following:*

(3.$a$) *minimal type: $\{I, -I\}$;*

(3.$b$) *for some vector $p$,*

$$\{I, -I, \mathrm{Ref}_0(p), \mathrm{Rot}_\pi(p)\};$$

(3.$c$) *for some orthogonal $p_1, p_2, p_3$,*

$$\{I, -I, \mathrm{Ref}_0(p_1), \mathrm{Ref}_0(p_2), \mathrm{Ref}_0(p_3), \mathrm{Rot}_\pi(p_1), \mathrm{Rot}_\pi(p_2), \mathrm{Rot}_\pi(p_3)\};$$

(3.$d$) *for some orthogonal $p_1, p_2, p_3$,*

$$\{I, -I\} \cup (\mathrm{Rot}_\theta(p_3) \cup \mathrm{Ref}_\theta(p_3)) \cup (\mathrm{Rot}_\pi(p_3) \cup \mathrm{Ref}_0(p_3)) \cup \bigcup_{q \in \mathrm{span}_{\mathbb{R}}\{p_1, p_2\}} (\mathrm{Rot}_\pi(q) \cup \mathrm{Ref}_0(q));$$

(3.$e$) *maximal type: $O(3)$.*

PROOF: Recall that, under (5.19), the symmetry group $G_H$ is conjugate to $\cap_{x>0} G(\Pi_x)$. By Proposition 5.2, $G(\Pi_x)$ can only be of the forms (5.15), (5.16) and (5.17). The proof is now split into the following cases.

Case 1: for some $x > 0$, $G(\Pi_x)$ has the form (5.17). Since the intersection with some other $G(\Pi_{x'})$ can only reduce the group, $\cap_{x>0} G(\Pi_x)$ can be only of types (3.$a$), (3.$b$) or (3.$c$), where (3.$b$) is a consequence of intersecting (5.17) with some appropriate (5.17).

Case 2: all $G(\Pi_x)$ are the same and have the form (5.16). This gives type (3.$d$).

Case 3: there are $x_1 \neq x_2$ and two different $G(\Pi_{x_1})$ and $G(\Pi_{x_2})$ such that both have the form (5.16). Let $p_{1,1}, p_{1,2}, p_{1,3}$ and $p_{2,1}, p_{2,2}, p_{2,3}$ be the corresponding orthonormal vectors in (5.16). For $G(\Pi_{x_1})$ and $G(\Pi_{x_2})$ to be different, the corresponding axis $\text{span}_{\mathbb{R}}\{p_{1,3}\}$ and $\text{span}_{\mathbb{R}}\{p_{2,3}\}$ have to be different. Then, $G(\Pi_{x_1}) \cap G(\Pi_{x_2})$ is of type (3.$a$), (3.$b$) or (3.$c$), if, respectively, $p_{2,3} \notin \text{span}_{\mathbb{R}}\{p_{1,1}, p_{1,2}\}$, $p_{2,3} \in \text{span}_{\mathbb{R}}\{p_{1,1}, p_{1,2}\} \backslash (\text{span}_{\mathbb{R}}\{p_{1,1}\} \cup \text{span}_{\mathbb{R}}\{p_{1,2}\})$, $p_{2,3} \in \text{span}_{\mathbb{R}}\{p_{1,1}\} \cup \text{span}_{\mathbb{R}}\{p_{1,2}\}$. Regarding type (3.$b$), further intersections may only result in type (3.$b$) again or type (3.$a$). Regarding type (3.$c$), one is thus back to Case 1.

Case 4: For all $x > 0$, $G(\Pi_x)$ has the form (5.15). This gives type (3.$e$). $\Box$

We now provide examples of OFBMs of the types (3.$b$), (3.$c$) and (3.$d$), thereby showing that all the types described in Theorem 5.2 are non-empty.

**Example 5.3** (Type (3.$b$)) Consider the OFBM with spectral representation parameters $A := I$ and

$$D = \begin{pmatrix} d & 0 & 0 \\ 1 & d & 0 \\ 0 & 0 & d \end{pmatrix}.$$

By Theorem 3.1, we may assume that the positive definite conjugacy associated with $G(B_H)$ is $W = I$. Observe that

$$\Pi_x = x^{-D} x^{-D^*} = x^{-2d} \begin{pmatrix} 1 & -\log(x) & 0 \\ -\log(x) & \log^2(x) + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Due to the block-diagonal shape of $x^{-D} x^{-D^*}$, it suffices to focus on its $2 \times 2$ upper left block. Consider $x = e^{-1}, e$. The associated $2 \times 2$ blocks, i.e.,

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

have pairwise different eigenvalues. Moreover, they do not share eigenvectors, since otherwise they would commute. As a consequence, by Proposition 5.2, they are of the form (3.$b$) with $p = (0, 0, 1)'$.

**Example 5.4** (Type (3.$c$)) Consider the OFBM with spectral representation parameters

$$D = \text{diag}(d_1, d_2, d_3), \ \ A = \text{diag}(a_1, a_2, a_3), \tag{5.20}$$

where $d_i \neq d_j$, $i \neq j$. Then, $AA^* = \text{diag}(|a_1|^2, |a_2|^2, |a_3|^2) = \Re(AA^*)$, $\Im(AA^*) = 0$ and $\Pi_x = \text{diag}(x^{-2d_1}, x^{-2d_2}, x^{-2d_3})$. This yields

$$G_H = \{I, -I, \text{Ref}_0(e_1), \text{Ref}_0(e_2), \text{Ref}_0(e_3), \text{Rot}_\pi(e_1), \text{Rot}_\pi(e_2), \text{Rot}_\pi(e_3)\}, \tag{5.21}$$

where $e_1 = (1, 0, 0)'$, $e_2 = (0, 1, 0)'$, $e_3 = (0, 0, 1)'$.

**Example 5.5** (Type (3.$d$)) Consider the OFBM with spectral representation parameters

$$D = \mathrm{diag}(d_1, d_1, d_3), \;\; A = \mathrm{diag}(a_1, a_2, a_3), \tag{5.22}$$

where $d_1 \neq d_3$. Then, $AA^* = \mathrm{diag}(|a_1|^2, |a_2|^2, |a_3|^2) = \Re(AA^*)$, $\Im(AA^*) = 0$ and $\Pi_x = \mathrm{diag}(x^{-2d_1}, x^{-2d_1}, x^{-2d_3})$. This yields

$$G_H = \{I, -I\} \cup (\mathrm{Rot}_\theta(e_3) \cup \mathrm{Ref}_\theta(e_3)) \cup (\mathrm{Rot}_\pi(e_3) \cup \mathrm{Ref}_0(e_3)) \cup \bigcup_{q \in \mathrm{span}\{e_1,e_2\}} (\mathrm{Rot}_\pi(q) \cup \mathrm{Ref}_0(q)), \tag{5.23}$$

where $e_1, e_2, e_3$ are as in Example 5.4.

We now extend Theorem 5.2 to the general case of OFBMs which are not necessarily time reversible, i.e., we drop the assumption (5.19). From the perspective of the structural result provided by Theorem 3.1, the lack of time reversibility manifests itself as an additional constraint which may reduce the symmetry group, and even generate a new type, as seen in the next theorem.

**Theorem 5.3** *Consider an OFBM given by the spectral representation (1.3), and suppose that the matrix $A$ satisfies the assumption (2.4). Then, its symmetry group $G_H$ is conjugate by a positive definite matrix $W$ to the ones described in Theorem 5.2, plus the following:*

(3.$f$) *for some vector $p$,*

$$\{I, -I, \mathrm{Ref}_0(p), \mathrm{Rot}_\pi(p), \mathrm{Ref}_\theta(p), \mathrm{Rot}_\theta(p)\}.$$

Proof: If $\Pi_I = 0$, then $G(\Pi_I) = O(n)$. So, assume $\Pi_I \neq 0$. Since $\Pi_I \in so(3)$, then there exists $S_I := (p_1\, p_2\, p_3) \in O(3)$ such that $\Pi_I = S_I \mathrm{diag}(L_s, 0) S_I^*$, where $L_s \neq 0$ has the form (2.5). Therefore, by Theorem 2.1, we have

$$G(\Pi_I) = S_I \mathrm{diag}(SO(2), \pm 1) S_I^*. \tag{5.24}$$

For a matrix of the form $O\mathrm{diag}(SO(2), \pm1)O^* = OU\mathrm{diag}(e^{i\theta}, e^{-i\theta}, \pm 1)U^*O^*$, $\theta \neq 0$, where $U = \mathrm{diag}(U_2, 1)$ and $U_2$ is as in (4.14), only the eigenvalue 1 (or $-1$) is associated with a purely real eigenvector. As a consequence, the intersection between $G(\Pi_I)$ and one of the subgroups in Theorem 5.2 can only be different from (3.$a$) if the eigenspace associated with 1 or $-1$ (i.e., the space generated by the third column of $S_I$) coincides with $\mathrm{span}_{\mathbb{R}}\{p_3\}$.

Thus, by intersecting (5.24) with either (3.$b$) or (3.$c$), we obtain (3.$b$). Moreover, by intersecting (5.24) with either (3.$d$) or (3.$e$), we obtain (3.$f$). $\Box$

**Example 5.6** (Type (3.$f$)) Analogously to Example 5.1, consider an OFBM with parameters

$$D = dI, \quad \Re(AA^*) = I, \quad \Im(AA^*) = \mathrm{diag}(L, 0), \quad L \in so(2)\backslash\{0\},$$

where $d$ is real. Then, $G(\Pi_x) = O(3)$ and $G(\Pi_I)$ is as in (5.24).

Theorems 5.2 and 5.3 stand in contrast with Theorem 5.1 in that they show the much greater wealth of possible symmetry groups in dimension 3 as compared to dimension 2. In a certain sense, this enhances the claim of Theorem 4.2 in that, notwithstanding the increasing complexity of the possible symmetry structures as dimension increases, minimal type symmetry groups remain the topologically general case for *any* dimension.

We now provide the tangent spaces and exponent sets for each symmetry group with non-trivial tangent space. The proof is along the lines of that for Proposition 5.1.

**Proposition 5.3** *Under the assumptions of Theorem 5.3, for the symmetry groups associated with non-trivial tangent spaces, the tangent spaces, commuting exponents $H_0$ and sets of exponents have the form:*

*(3.d) for some orthonormal $p_1$, $p_2$, $p_3$, and the associated matrix $S := (p_1\, p_2\, p_3)$,*

$$T(G_H) = WS\mathrm{diag}(so(2), 0)S^* W^{-1},$$

$$H_0 = WSU\mathrm{diag}(h_1, \overline{h}_1, h_2)U^* S^* W^{-1},$$

$$\mathcal{E}(B_H) = WSU(\mathrm{diag}(h_1, \overline{h}_1, h_2) + \mathrm{diag}(so(2), 0))S^* W^{-1},$$

*where $U = \mathrm{diag}(U_2, 1)$ and $U_2$ is as in (4.14), and $h_1 \in \mathbb{C}$, $h_2 \in \mathbb{R}$;*

*(3.e)*

$$T(G_H) = T(SO(3)) = Wso(3)W^{-1},$$

$$H_0 = h_0 I,$$

$$\mathcal{E}(B_H) = h_0 I + Wso(3)W^{-1};$$

*(3.f) the same as for (3.d).*

Proof: For type (3.$d$), just note that $T(G_H) = T(\mathrm{Rot}_\theta(p_3)) = WS\mathrm{diag}(so(2), 0)S^* W^{-1}$, from which $H_0$ and $\mathcal{E}(B_H)$ promptly follow. The same argument holds for type (3.$f$).

The case of type (3.$e$) is straightforward, since $T(G_H) = T(SO(3))$. $\Box$

**Remark 5.4** In general dimension $n$, there are no additional difficulties in describing the structure of groups $G(\Pi)$ for a *fixed* symmetric matrix $\Pi$. Equivalently, one can generalize Proposition 5.2 to the context of dimension $n$ without much effort. Nevertheless, it is cumbersome to describe the structure of intersections $G(\Pi_1) \cap G(\Pi_2)$, which is needed for the full characterization of symmetry groups $G_H$ as in (3.13) and (3.17). At this point, a full description of symmetry groups in general dimension $n$ is an open question.

**Remark 5.5** The classification given in Theorem 5.1 stands in contrast with the fact that $SO(n)$ cannot be a symmetry group for $\mathbb{R}^n$-valued random vectors (Billingsley (1966)). In particular, $SO(2)$ is not a maximal element of its equivalence class of subgroups in the sense of Meerschaert and Veeh (1995), p. 2 (not to be confused with the symmetry group of maximal type in Theorem 5.1). However, it turns out that Billingsley's result is actually *almost* true for OFBMs, and more generally, proper zero mean Gaussian processes. In other words, for the latter class of processes, $SO(n)$ can only be a symmetry group when $n = 2$ (cf. Theorem 5.3). Indeed, without loss of generality, assume $W = I$. Then, it suffices to show that $SO(n) \subseteq G(X)$ implies that $O(n) = G(X)$ when $n \geq 3$. However, the latter equivalence is a consequence of Proposition A.1 in the appendix.

## 6 On integral representations of OFBMs with multiple exponents

In this section, we show that when an OFBM has multiple exponents, the matrix $A$ in (1.3) can be chosen the same, no matter what matrix exponent is used in the parametrization. We also show that, by contrast, such invariance of the parametrization does *not* hold for the so-called time domain representation of OFBM.

We first consider the latter point. Under (1.2) and $\Re(h) \neq 1/2$ for any eigenvalue $h$ of $H$, the OFBM $\{B_H(t)\}_{t\in\mathbb{R}}$ also admits an integral representation in the time domain, i.e.,

$$\{B_H(t)\}_{t\in\mathbb{R}} \stackrel{\mathcal{L}}{=} \Big\{ \int_{\mathbb{R}} (((t-u)_+^{H-\frac{1}{2}I} - (-u)_+^{H-\frac{1}{2}I})M_+ + ((t-u)_-^{H-\frac{1}{2}I} - (-u)_-^{H-\frac{1}{2}I})M_-)B(du) \Big\}_{t\in\mathbb{R}}, \tag{6.1}$$

where $M_+$, $M_- \in M(n,\mathbb{R})$, and $\{B(u)\}_{u\in\mathbb{R}}$ is a vector-valued process consisting of independent Brownian motions and such that $EB(du)B(du)^* = du$ (Didier and Pipiras (2010)). The following example shows that, in general, the matrix parameters $M_+$, $M_-$ cannot be chosen independently of the exponent.

**Example 6.1** Consider a bivariate OFBM $B_H$ with the time domain representation (6.1), where $D = dI$, $d \in (-1/2, 1/2)\backslash\{0\}$ (or $H = hI$, $h \in (0,1)\backslash\{1/2\}$), $M_+ = O \in SO(2)$ and $M_- = I$. Since rotation matrices commute, it follows directly from (6.1) that

$$SO(2) \subseteq G_H. \tag{6.2}$$

The relation (6.2) implies that $T(SO(2)) = so(2) \subseteq T(G_H)$. Hence, in view of (1.5),

$$H + L_c, \quad c \in \mathbb{R}, \tag{6.3}$$

are the exponents of the OFBM $B_H$, where $L_c \in so(2)$ is given in (2.5). Thus, the OFBM $B_H$ has the time domain representation

$$\{B_H(t)\}_{t\in\mathbb{R}} \stackrel{\mathcal{L}}{=} \Big\{ \int_{\mathbb{R}} (((t-u)_+^{D+L_c} - (-u)_+^{D+L_c})M_+ + ((t-u)_-^{D+L_c} - (-u)_-^{D+L_c})M_-)B(du) \Big\}_{t\in\mathbb{R}}, \tag{6.4}$$

where $M_+ = M_+(c)$, $M_- = M_-(c)$. We want to show that one cannot generally take the original parameters $M_+ = O$, $M_- = I$ in the representation (6.4).

Arguing by contradiction, suppose that $M_+ = O$, $M_- = I$ in (6.4) lead to the same OFBM for any $c \in \mathbb{R}$. In the spectral domain, these processes have the representation

$$\int_{\mathbb{R}} \frac{e^{itx}-1}{ix}(x_+^{-D-L_c}A + x_-^{-D-L_c}\overline{A})\widetilde{B}(dx), \tag{6.5}$$

where $\widetilde{B}(dx)$ is as in (1.3), and

$$A = \frac{1}{\sqrt{2\pi}}\Gamma(D+L_c+I)(e^{-i\pi(D+L_c)/2}O + e^{i\pi(D+L_c)/2}I)$$

(see Theorem 3.2 and its proof in Didier and Pipiras (2010)). Note that $x^{-L_c}$ commutes with $A$, $\overline{A}$ and hence $(-L_c)$ can be removed from the exponents of $x_+$, $x_-$ in (6.5). Then, if (6.5) is the same process for all $c \in \mathbb{R}$, the matrix

$$\begin{aligned}(2\pi)AA^* &= \Gamma(dI+L_c+I)(e^{-i\pi(dI+L_c)/2}O + e^{i\pi(dI+L_c)/2}I)\cdot \\ &\quad \cdot(O^*e^{i\pi(dI+L_c^*)/2} + Ie^{-i\pi(dI+L_c^*)/2})\Gamma(dI+L_c+I)^* \\ &= \Gamma(dI+L_c+I)\Gamma(dI+L_c+I)^*(e^{-i\pi dI}O + e^{i\pi dI}O^* + e^{-i\pi L_c} + e^{i\pi L_c}) \end{aligned} \tag{6.6}$$

does not depend on $c$. Note that

$$\Gamma(dI + L_c + I)\Gamma(dI + L_c + I)^* = U_2 \mathrm{diag}(|\Gamma(d + ic + 1)|^2, |\Gamma(d + ic + 1)|^2)U_2^* = |\Gamma(d + ic + 1)|^2 I,$$

where $U_2$ is as in (4.14) and $\Gamma(d+ic+1)$ is the univariate Gamma function evaluated at $d+ic+1 \in \mathbb{C}$. Writing $O = U_2\mathrm{diag}(e^{i\beta}, e^{-i\beta})U_2^*$, for some $\beta \in (0, 2\pi)\backslash\{\pi\}$, the matrix (6.6) becomes

$$U_2\mathrm{diag}(f(d, c, \beta), \overline{f(d, c, \beta)})U_2^*,$$

where

$$f(d, c, \beta) := |\Gamma(d + ic + 1)|^2 \Big(e^{-i\pi d}e^{i\beta} + e^{i\pi d}e^{-i\beta} + e^{\pi c} + e^{-\pi c}\Big).$$

However, the function $f(d, c, \beta)$ does depend on $c$, as can be easily verified (contradiction).

The following result shows that, for a given OFBM, one can take the same parameter $A$ in the spectral representation (1.3) for all exponents $H$ of the OFBM in question.

**Theorem 6.1** *Let $B_H$ be an OFBM having the spectral representation (1.3). If $H_\lambda, H_\eta \in \mathcal{E}(B_H)$ and $A_\lambda$, $A_\eta$ are the two matrix parameters in (1.3) associated with $H_\lambda$, $H_\eta$, respectively, then*

$$A_\lambda A_\lambda^* = A_\eta A_\eta^*. \tag{6.7}$$

*In particular, one may choose the same matrix parameter $A$ in (1.3) for every choice of $H \in \mathcal{E}(B_H)$.*

Proof: It is enough to show (6.7) with a commuting exponent $H_\eta := H_0$ (see (5.10)) and the associated matrix $A_\eta := A_0$. For simplicity, let $H = H_\lambda$, $A = A_\lambda$. We know that

$$H - H_0 = D - D_0 =: \Delta \in W\mathcal{L}_0 W^{-1} = T(G_H),$$

where $\mathcal{L}_0 \subseteq so(n)$. We can thus write $\Delta = WLW^{-1}$ with $L \in so(n)$. The uniqueness of the spectral density of OFBM implies that, for $x > 0$,

$$x^{-D}AA^*x^{-D^*} = x^{-D_0}A_0A_0^*x^{-D_0^*},$$

or

$$x^{-(D_0+\Delta)}AA^*x^{-(D_0+\Delta)^*} = x^{-D_0}A_0A_0^*x^{-D_0^*}.$$

Since $D_0$ is a commuting exponent and $\Delta \in T(G_1)$, then $D_0$ and $\Delta$ commute. Hence,

$$x^{-D_0}x^{-\Delta}AA^*x^{-\Delta^*}x^{-D_0^*} = x^{-D_0}A_0A_0^*x^{-D_0^*}$$

and

$$x^{-\Delta}AA^*x^{-\Delta^*} = A_0A_0^*,$$

i.e.,

$$x^{-L}W^{-1}AA^*W^{-1}x^L = W^{-1}A_0A_0^*W^{-1}.$$

By differentiating with respect to $x$, we further obtain

$$L(W^{-1}AA^*W^{-1}) = (W^{-1}AA^*W^{-1})L,$$

that is, $L$ and $W^{-1}AA^*W^{-1}$ commute. Then,

$$W^{-1}AA^*W^{-1} = W^{-1}A_0A_0^*W^{-1}$$

or $AA^* = A_0A_0^*$. The last statement of the theorem follows from (6.7). $\square$

# A Auxiliary results on matrix commutativity

We begin by proving Lemma 2.1. The argument draws upon Theorem 2.1.

Proof of Lemma 2.1: Since $\Gamma$ commutes with $A$, then it is diagonalizable over $\mathbb{C}$. Assume by contradiction that one of the eigenvalues of $\Gamma$, denoted $\lambda_1$, is different from all the others. Then, for an associated eigenvector, denoted $v_1$, the one-dimensional eigenspace $\mathrm{span}_{\mathbb{C}}(v_1)$ is by Theorem 2.1 an eigenspace of both $\Gamma$ and $A$. If $\Gamma$ also commutes with $B$, then by the same argument $\mathrm{span}_{\mathbb{C}}(v_1)$ must also be an eigenspace of $B$, i.e., $A$ and $B$ share an eigenvector (contradiction). $\square$

The next proposition is used in Remark 5.5. It shows that, for $n \geq 3$, the group $SO(n)$ is so rich that only a matrix which is a multiple of the identity can contain it in its centralizer. For $n = 2$, one needs to consider instead the entire orthogonal group.

**Proposition A.1** *Let $\Gamma \in M(n, \mathbb{R})$. Then, $\Gamma = \lambda I$, $\lambda \in \mathbb{R}$, if one of the following assumptions holds:*

*(i) for $n = 2$, if $\mathcal{C}(\Gamma) \supseteq O(n)$;*

*(ii) for $n \geq 3$, if $\mathcal{C}(\Gamma) \supseteq SO(n)$.*

Proof: We only prove $(ii)$. Without loss of generality, assume $n$ is even. We can take some $O_1 \in SO(n)$ such that

$$O_1 = PU\mathrm{diag}(e^{i\theta_1}, e^{-i\theta_1}, e^{i\theta_2}, e^{-i\theta_2}, ..., e^{i\theta_{n/2}}, e^{-i\theta_{n/2}})U^* P^*, \quad \theta_1, \theta_2, ..., \theta_{n/2} \in (0, 2\pi)\backslash\{\pi\},$$

where $P \in O(n)$, $U = \mathrm{diag}(U_2, \ldots, U_2)$, and $U_2$ is as in expression (4.14). Now choose another $O_2 \in SO(n)$ such that

$$O_2 = QU\mathrm{diag}(e^{i\mu_1}, e^{-i\mu_1}, e^{i\mu_2}, e^{-i\mu_2}, ..., e^{i\mu_{n/2}}, e^{-i\mu_{n/2}})U^* Q^*, \quad \mu_1, \mu_2, ..., \mu_{n/2} \in (0, 2\pi)\backslash\{\pi\},$$

where no column vector of $Q \in O(n)$ can be written as a linear combination of *two* column vectors of $P$ (this can be obtained by slightly perturbing the matrix $P$, since $n \geq 3$). By considering the matrices $PU$ and $QU$, we can see that

$$p_{2j-1} \pm i p_{2j}, \quad q_{2k-1} \pm i q_{2k}, \quad j, k = 1, ..., \frac{n}{2}, \tag{A.1}$$

are eigenvectors of $O_1$ and $O_2$, respectively.

We claim that $O_1$ and $O_2$ have no (complex) eigenvectors in common. In fact, if they did, then for some pair $j, k$ there would exist $z \in \mathbb{C}$ such that

$$z(p_{2j-1} + i p_{2j}) = q_{2k-1} + i q_{2k}$$

(without loss of generality, we only take + signs in (A.1)). Thus,

$$\Re(z) p_{2j-1} - \Im(z) p_{2j} = q_{2k-1}, \quad \Re(z) p_{2j} + \Im(z) p_{2j-1} = q_{2k},$$

which contradicts our assumption on $Q$.

Thus, if some $\Gamma \in GL(n)$ commutes with $O_1$ and $O_2$, then by Lemma 2.1 it must have the form $\Gamma = \lambda I$, $\lambda \in \mathbb{R}$. $\square$

## B Other results on the convergence of eigenvalues and eigenvectors

The first lemma shows that the convergence of matrices implies the convergence of the eigenvalues. The second shows that, under more stringent assumptions, it also implies the convergence of the eigenspaces.

**Lemma B.1** *Let $\{A_k\}_{k\in\mathbb{N}}, A_0 \in M(n,\mathbb{C})$, and assume that $A_k \to A_0$. Then, the eigenvalues of $A_k$ converge to those of $A_0$, i.e., one can form a sequence $\{(\lambda_k^1,\ldots,\lambda_k^n)\}_{k\in\mathbb{N}} \subseteq \mathbb{C}^n$ of eigenvalues of $A_k$, $k \in \mathbb{N}$, such that*

$$(\lambda_k^1,\ldots,\lambda_k^n) \to (\lambda_0^1,\ldots,\lambda_0^n), \tag{B.1}$$

*where the vector on the right-hand side of (B.1) is made up of eigenvalues of $A_0$.*

Proof: Define the class of polynomials

$$f_k(\lambda) = \det(A_k - \lambda I), \quad \lambda \in \mathbb{C}, \quad k \in \mathbb{N}\cup\{0\}.$$

Since $A_k \to A_0$, then by the continuity of the determinant function, $f_k \to f_0$ pointwise. Thus, the roots of the polynomials $f_k$ must converge pointwise to those of $f_0$. □

**Lemma B.2** *Let $\{A_k\}_{k\in\mathbb{N}}, A_0 \in M(n,\mathbb{C})$ such that $A_0$ has pairwise different eigenvalues. Assume that $A_k \to A_0$. Then, for the sequence of eigenvalues in (B.1), there exists a sequence of conjugacies $\{P_k\}_{k\in\mathbb{N}} \subseteq GL(n,\mathbb{C})$ such that*

$$A_k = P_k \mathrm{diag}(\lambda_k^1,\ldots,\lambda_k^n) P_k^{-1}$$

*and $P_k \to P$ for some $P \in GL(n,\mathbb{C})$. Moreover, the columns of the limiting matrix $P$ are, in fact, eigenvectors of $A_0$.*

Proof: For $j = 1,\ldots,n$, take an associated eigenvector $p_k^j \in S^{n-1}$, and also denote by $p_0^j \in S^{n-1}$ an eigenvector of $A_0$ associated with the eigenvalue $\lambda_0^j$. Assume by contradiction that $\{p_k^j\}$ converges neither to $p_0^j$ nor to $-p_0^j$, i.e., there exists $\varepsilon_0 > 0$ and a subsequence $\{p_{k'}^j\}$ such that

$$\left\|p_{k'}^j - p_0^j\right\| \geq \varepsilon_0, \quad \left\|p_{k'}^j - (-p_0^j)\right\| \geq \varepsilon_0.$$

Since $\{p_{k'}^j\} \subseteq S^{n-1}$, one can extract a further subsequence $\{p_{k''}^j\}$ which is convergent, i.e.,

$$p_{k''}^j \to p'' \in S^{n-1}, \quad p'' \notin \{p_0^j, -p_0^j\}.$$

Therefore,

$$A_{k''} p_{k''}^j = \lambda_{k''}^j p_{k''}^j,$$

where

$$A_{k''} p_{k''}^j \to A_0 p'', \quad \lambda_{k''}^j p_{k''}^j \to \lambda_0^j p''.$$

Thus, $p'' \in S^{n-1}$ is an eigenvector of $A_0$ associated with the eigenvalue $\lambda_0^j$. However, since the eigenvalues of $A_0$ are pairwise different, then $p'' \in \{p_0^j, -p_0^j\}$ (contradiction). □

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