More trig. identities: Product & Sum:
(chapter 6.5)

Recall:

$$\left.\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \end{aligned}\right\} \text{ add these:}$$

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = \sin\alpha\cos\beta + \cancel{\cos\alpha\sin\beta} + \sin\alpha\cos\beta - \cancel{\cos\alpha\sin\beta}$$

$$= 2\sin\alpha\cos\beta$$

$$\Rightarrow \boxed{\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)}.$$

similarly:

$$\left.\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \end{aligned}\right\} \text{ add these}$$

$$\boxed{\cos\alpha\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)}$$

and two more obtained in a similar fashion:

$$\boxed{\begin{aligned} \sin\alpha\sin\beta &= \frac{1}{2}\left(\sin(\alpha+\beta) - \sin(\alpha+\beta)\right). \\ \cos\alpha\sin\beta &= \frac{1}{2}\left(\sin(\alpha+\beta) - \sin(\alpha-\beta)\right) \end{aligned}}$$

Ex: Find the exact value of the product:

$$\cos(\underbrace{67.5^\circ}_{\alpha}) . \sin(\underbrace{112.5^\circ}_{\beta})$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right) =$$

$$= \frac{1}{2}\left(\sin(67.5^\circ + 112.5^\circ) - \sin(67.5^\circ - 112.5^\circ)\right)$$

$$= \frac{1}{2}\left(\sin 180^\circ - \sin(-45^\circ)\right)$$

$$= \frac{1}{2}\left(0 + \sin(45^\circ)\right).$$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}.$$

Conditional trig. equations

Chapter 6.2

Def: conditianal equation: these are not identities but have at least one solution.

Ex: $\sin x = 0$ for $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

Examples:

① $\cos x = 2$
② $\sin x = 2.$

} no solution! recall for $\forall x$: $-1 \leq \cos x \leq 1$, $-1 \sin x \leq 1.$

③ $\cos x = 1$ $\qquad x = 2k\pi$

④ $\cos x = 0$ $\qquad x = \frac{\pi}{2} + k\pi$

⑤ $\sin x = 1$ $\qquad x = \pi/2 + 2k\pi$

⑥ $\sin x = 0.$ $\qquad x = k\pi.$

⑦ $\sin x = 1/2$ $\qquad x = \frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi.$

⑧ $\sin x = 1/2$ for $x \in \left[\pi/2, \frac{3\pi}{2}\right].$ $\qquad x = \frac{5\pi}{6}$ (only)!

Solving more complicated equations:

General strategy:

- simplify complicated expressions using identities
- try to ~~get~~ rewrite those with only one trig. function.
- check if equation is of quadratic type – then use quadratic formula
- for equations involving multiple angles solve for the multiple angle first. eg:

$\sin 2u = \frac{\sqrt{2}}{2} \Rightarrow 2u = x.$

solve

$\sin x = \frac{\sqrt{2}}{2} \Rightarrow x = 45^\circ, 135^\circ, \dots$

but $x = 2u \Rightarrow u = \frac{45^\circ}{2}, \frac{135^\circ}{2}, \dots$

Examples:

(1) $6\cos^2\left(\frac{x}{2}\right) - 7\cos\left(\frac{x}{2}\right) + 2 = 0.$

Let $\cos\left(\frac{x}{2}\right) = u$

$6u^2 - 7u + 2 = 0$

$(2u - 1)(3u - 2) = 0$

$\Rightarrow 2u - 1 = 0$ or $3u - 2 = 0$

$u = 1/2$ or $u = 2/3.$

$\left(\alpha = \frac{\pi}{3} + 2k\pi, \ \frac{5\pi}{3} + 2k\pi\right).$

$u = \cos\frac{x}{2} \Rightarrow \cos\frac{x}{2} = \frac{1}{2} \Rightarrow \cos\alpha = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}, \frac{5\pi}{3}, \dots$

$\alpha = \frac{x}{2} \Rightarrow x = 2\alpha = \frac{2\pi}{3}, \frac{10\pi}{3}, \dots$

similarly for $\cos\frac{x}{2} = \frac{2}{3}$

Ex : $\tan 3x + 1 = \sqrt{2} \sec 3x$, for $x \in [0, 2\pi)$

we make our way towards the identity $\sec^2 \alpha = \tan^2 \alpha + 1$.
$\Rightarrow$ square both sides :

$$\left(\tan(3x)+1\right)^2 = \left(\sqrt{2}\sec(3x)\right)^2$$

also let $\underline{3x = u}$ substitute :

$$(\tan u + 1)^2 = 2\sec^2 u.$$

$$\underbrace{\tan^2 u + 1}_{\sec^2 u.} + 2\tan u = 2\sec^2 u.$$

$2\tan u = \sec^2 u$ or $-\tan^2 u - 1 + 2\tan u = 0.$

$$\tan^2 u - 2\tan u + 1 = 0.$$

$$(\tan u - 1)^2 = 0$$

$$\tan u = 1.$$

$$u = \frac{\pi}{4} + 2k\pi.$$

$$u = 3x \Rightarrow x = \frac{u}{3} = \frac{\pi}{12} + \frac{2k\pi}{3}$$

recall $x \in [0, 2\pi)$

$\Rightarrow$ ...

because we took square on each side, check if the roots we obtained are correct!

Ex: solve:

$$\sin 2x \cos x - \cos 2x \sin x = 1/2$$

let $2x = \alpha$
$x = \beta$

we get: $\underbrace{\sin\alpha \cos\beta - \cos\alpha \sin\beta}_{\sin(\alpha-\beta)} = +1/2.$

$\Rightarrow \sin(\alpha-\beta) = +1/2$ but $\alpha = 2x$, $\beta = x$.

$\sin(\alpha-\beta) = \sin(2x - x) = \sin x$

$\Rightarrow \sin x = +1/2.$

$x = \frac{\pi}{6} + 2k\pi$ or $\frac{5\pi}{6} + 2k\pi$.

alternatively. (much longer)...

$\sin 2x = 2\sin x \cos x$

$\cos 2x = 1 - 2\sin^2 x$

$\Rightarrow 2\sin x(\cos x)^2 - (1 - 2\sin^2 x)\cos x = 1/2.$

$2\sin x \underbrace{\cos^2 x}_{1-\sin^2 x} \cancel{*} -1 + 2\sin^2 x \cos x = 1/2.$

$2\sin x - 2\sin^3 x + 2\sin^2 x \cos x = 3/2.$ $.1/2.$

$\sin x - \sin^3 x + \sin^2 x \cos x = 3/4.$

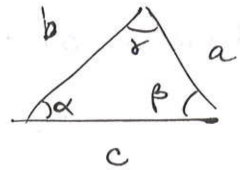
use the identities here.

$\sin x(1 + \cos x \sin x \cancel{+} (\sin^2 x)) = \sin x(\cos^2 x + \cos x \sin x) = \sin x \cos x(\ldots$
$(\cos x + \sin x)$

Law of sines
§7.1

Thm: The law of sines: the ratio of the sine of an angle and the length of the side opposite to it (the angle) is the same for each angle of the Δ: i.e.

If we have the triangle:

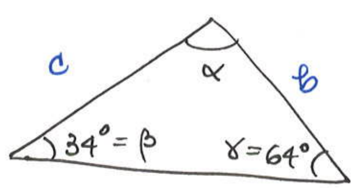


then: $$\boxed{\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}}$$

equivallently: $$\boxed{\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}}.$$

Solving the triangle means finding $a, b, c, \alpha, \beta, \gamma$ when we are given 3 sides, 2 sides & an angle, 2 angles & a side.
SSS SAS & SSA ASA (AAS).

Example ① (two angles and an included side). - ASA



$a = 5.3$

$\alpha = ?$

$b = ?$

$c = ?$

$$\begin{aligned}\alpha &= 180^\circ - \beta - \gamma \\ &= 180^\circ - 34^\circ - 64^\circ \\ &= \underline{82^\circ}.\end{aligned}$$

$$\frac{\sin\alpha}{a} = \frac{\sin 82^\circ}{5.3} = \frac{\sin 34^\circ}{b} \Rightarrow b = \frac{\sin 34^\circ (5.3)}{\sin 82^\circ} \sim \underline{3.}$$

similarly $\underline{c \sim 4.8.}$

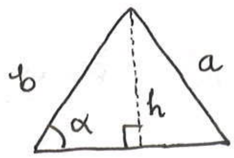
② Two sides and an angle (non-included). i.e. angle is not between them. (SSA)

We know α, a, b

The possible cases:

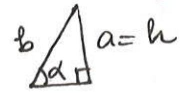
there exist: 0, 1 or 2 triangles with the above α, a, b.

consider:

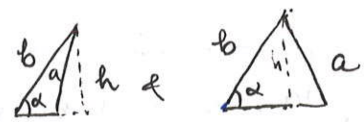


$\frac{h}{b} = \sin\alpha$, because $\frac{\sin\alpha}{h} = \frac{\sin 90^\circ}{b} = \frac{1}{b}$

case 1 if $a < h \Rightarrow$ no such Δ ($a \geq h$ being hypotenuse!)

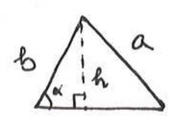


case 2 if $a = h \Rightarrow$ one right Δ



case 3 if $h < a < b \Rightarrow$ two triangles:

case 4 if $a \geq b \Rightarrow$ one Δ;



Ex: $\beta = 56.3^\circ$, $a = 8.3$, $b = 7.6$.

Solve the Δ:

(same as $\alpha = 56.3^\circ$, $a = 7.6$ & $b = 8.3$ in the notat. of the above discussion).

$a \leq b$.

Find h: $h = b\sin\alpha \Rightarrow h = (8.3)\sin 56.3 \approx 6.9$.

$h < a < b$ we are in case 3:

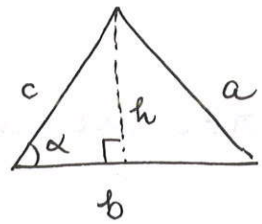
$\frac{\sin\beta}{8.3} = \frac{\sin\alpha}{7.6} \Rightarrow \sin\beta \approx 0.9 \Rightarrow \beta \approx 65.3^\circ$ or 114.7°

$\Rightarrow \gamma \approx 9^\circ$ & $c \approx 1.4$.

or $\gamma \approx 58.4^\circ$ & $c \approx 7.8$.

Area of triangle:

Consider the Δ:



area $= \frac{1}{2} bh$, but $h = c \sin\alpha$, becase $\frac{\sin 90°}{c} = \frac{\sin\alpha}{h} \Rightarrow$

$\frac{1}{c} = \frac{\sin\alpha}{h} \Rightarrow h = c\sin\alpha.$

$\Rightarrow \boxed{\text{area} = \frac{1}{2} bc \cdot \sin\alpha.}$

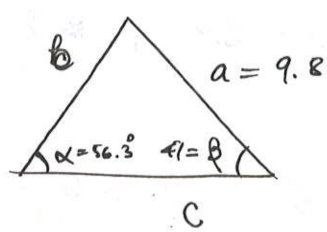
analogously $\boxed{\text{area} = \frac{1}{2} bc \sin\alpha = \frac{1}{2} ac \sin\beta = \frac{1}{2} ab \sin\gamma.}$

Ex: Find the area of the triangle with.

$\alpha = 56.3°$

$\beta = 41.2°$.

$a = 9.8$.



$\Rightarrow \frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$ note: $\gamma = 180° - 56.3° - 41.2° = 82.5°$

find c: $\Rightarrow c = \frac{a.\sin\gamma}{\sin\alpha} = \frac{(9.8)\sin(82.5°)}{\sin(56.3°)} = \frac{(9.8)(0.991)}{(0.832)} = 11.67$

area $= \frac{1}{2} ac \sin\beta = \frac{1}{2}(9.8)(11.67)\underbrace{(\sin 41.2°)}_{0.659} = \underline{37.68}$.

Law of cosines

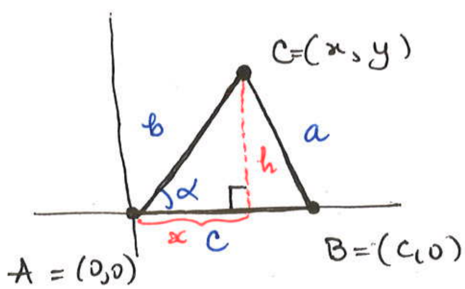
§ 7.2

Theorem: for a triangle with sides a, b, c & angles α, β, γ

$$\boxed{\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\alpha \\ b^2 &= a^2 + c^2 - 2ac\cos\beta \\ c^2 &= a^2 - b^2 - 2ab\cos\gamma \end{aligned}}$$

this is referred to as the law of cosines

Proof: Consider the triangle ABC:



$x = b\cos\alpha$ b/c $\cos\alpha = x/b$

$y = h = b\sin\alpha$ b/c $\sin\alpha = h/b$

} by definition of $\sin\alpha$ $\cos\alpha$.

$a = \sqrt{(x-c)^2 + (y-0)^2}$ from the distance formula

$=$ dist. b/w B & C.

$$\Rightarrow a^2 = ((b\cos\alpha) - c)^2 + (b\sin\alpha)^2$$

$$= b^2\cos^2\alpha + c^2 - 2bc\cos\alpha + b^2\sin^2\alpha$$

$$= b^2 + c^2 - 2bc\cos\alpha. \blacksquare$$

Ex: Solve the triangle:

③ (three sides)

$a = 8.2$ find $\alpha = ?$

$b = 3.7$ $\beta = ?$

$c = 10.8$ $\gamma = ?$

by law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\gamma$

$$\Rightarrow c^2 - a^2 - b^2 = -2ab\cos\gamma$$

$$\cos\gamma = \frac{c^2 - a^2 - b^2}{-2ab}$$

$$\cos\gamma \simeq -0.5885$$

$$\Rightarrow \gamma \approx 126.1^\circ$$

by law of sines: $\frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \Rightarrow \sin\beta = 0.2768$

$$\Rightarrow \beta \approx 16.1^\circ$$

$$\Rightarrow \alpha = 180^\circ - \beta - \gamma = 37.8^\circ.$$

④ Solve the triangle (two sides & included angle)

Ex: $a = 19.2$

$b = 37.6$

$\gamma = 68^\circ$ (angle b/w them).

by cosines law: $c^2 = (19.2)^2 + (37.6)^2 - 2.(19.2)(37.6)\cos 68$

$$= 1241.5$$

$$\Rightarrow c \approx 35.2.$$