Ch. 7.3: Vectors

Johns Hopkins University

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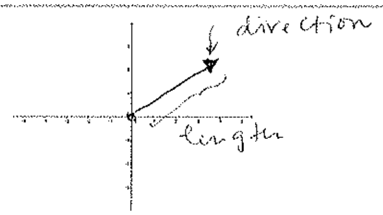
Vectors(1)

Definition (Vector)

A vector A quantity with magnitude and direction. The magnitude is the length of the vector and the direction is indicated by the position of the vector and arrow head on the end.

Example

Acceleration



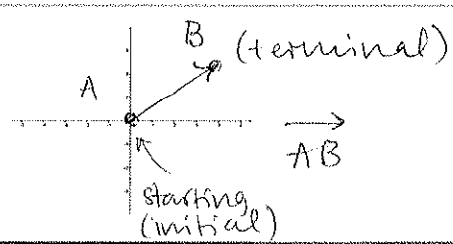
Vectors(2)

Definition

The starting point of a vector is the **initial point** and the end point is the **terminal point**.

Example

Vector starting at A and ending at B is denoted by $\vec{AB}$.



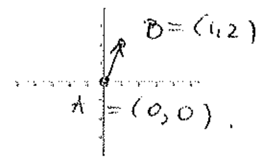
Vectors - N.B.

Remark (1)

We usually deal with vectors in starting position, that is the initial point is the origin, i.e. $A = (0,0)$. For these vectors we only give the terminal point.

Example

The vector $\vec{AB}$, with initial point $A = (0,0)$ and terminal point $B = (1,2)$ is denoted by $\langle 1,2 \rangle$.



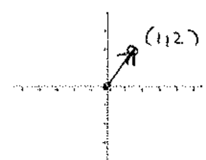
Vectors(3)

Definition

Magnitude of a vector is the length of the vector. The magnitude of the vector $\vec{AB}$ is denoted by $|\vec{AB}|$.

Question

What is the magnitude of the vector $\langle 1,2 \rangle$.



(x_1, y_1) (x_2, y_2)

dist. b/w $A = (0,0)$ & $B = (1,2)$

dist $= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(-1)^2 + (-2)^2}$

$= \sqrt{1+4} = \sqrt{5}$

Vectors - N.B.

Remark (2)

We denote by $\bar{AB}$ a line segment starting from A and ending at B and by $\vec{AB}$ the vector starting from A and ending at B.

Remark (3)

The vector $\vec{AB}$ the vector starting at A and ending at B is a different vector from $\vec{BA}$ the vector starting at B and ending at A.

Remark (4)

The vector $\vec{AA}$ the vector starting at A and ending at A is the **zero vector**.

Question

What's the magnitude of the zero vector?

(A) 0

(B) *depends what the starting point A is.*

(C) *we can't measure the magnitude.*

(D) *infinite*

Question

What can we saw about the magnitude of $\vec{AB}$ and $\vec{BA}$?

(A) $|\vec{AB}| = |\vec{BA}|$

(B) $|\vec{AB}| = -|\vec{BA}|$

(C) $|\vec{AB}| \geq |\vec{BA}|$

(D) *we can't always compare them.*

distance from A to B =
distance from B to A.

(going to school from home & back is the same dist.).

Scalar Multiplication

Definition

Let k be a real number. We refer to it as a **scalar**. Then the vector $k\vec{A}$ is the **scalar multiplication** of the vector $\vec{A}$ with k.

The magnitude of $k\vec{A}$ is the magnitude of $\vec{A}$ multiplied by $|k|$.

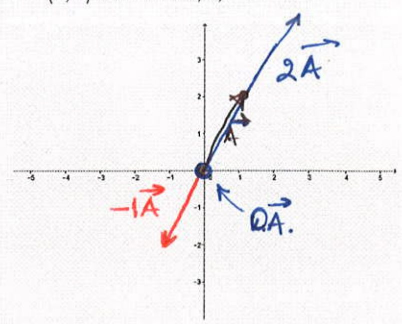
If $k > 0$ the direction of the vector $k\vec{A}$ is the same as the direction of $\vec{A}$, if $k < 0$ the direction of the vector $k\vec{A}$ is opposite to the direction of $\vec{A}$.

This is multiplying a vector by a real number - essentially we are stretching or shrinking the vector.

Scalar Multiplication

Example

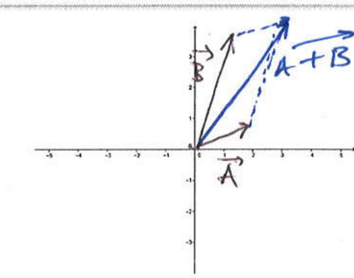
Sketch $k\vec{A}$, for $\vec{A} = \langle 1, 2 \rangle$ and $k = 0, 2, -1$.



Vector Addition - Geometric

Definition (Parallelogram Law)

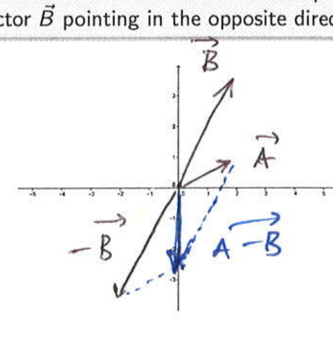
Consider two vectors $\vec{A}$ and $\vec{B}$ (where these have the same initial point!). Then $\vec{A} + \vec{B}$ is the vector beginning at their common initial point in the direction (and magnitude) the diagonal of the parallelogram with sides $\vec{A}$ and $\vec{B}$.



Vector Subtraction - Geometric

Definition

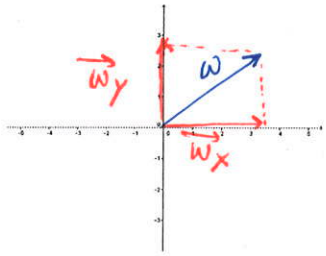
To find the difference of two vectors $\vec{A} - \vec{B}$ we compute $\vec{A} + (-\vec{B})$. Recall that $-\vec{B}$ is the vector $\vec{B}$ pointing in the opposite direction.



Horizontal and vertical components

Definition

We can think of every vector $\vec{w}$ as the sum of two vectors one lying on the x-axis and the other one on the y-axis. We refer to those as the **horizontal component** $\vec{w_x}$ and the **vertical component** $\vec{w_y}$.



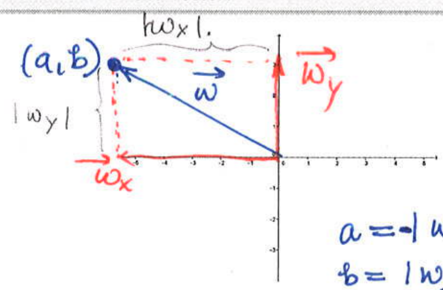
$\vec{w} = \vec{w_x} + \vec{w_y}$

by the law of prarellelog's.

Horizontal and vertical components

Definition

If the vector $\vec{w}$ is in standard position with horizontal component $\vec{w_x}$ and vertical component $\vec{w_y}$ then we say $\vec{w} = \langle \pm|w_x|, \pm|w_y| \rangle$. The signs depend on the direction of the horizontal and vertical components. This is the **component form** of the vector.



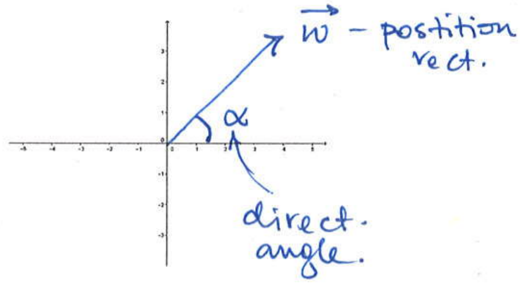
$a = -|w_x|$ b/c $a<0$

$b = |w_y|$. b/c $b>0$.

Horizontal and vertical components

Definition

If $\vec{w}$ is in standard position then we refer to it as the **position vector**. The angle between the positive x-axis and the position vector is called **direction angle**.



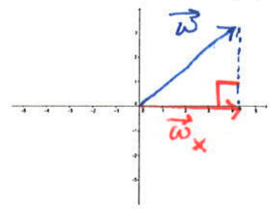
Finding horizontal and vertical components

Consider the right triangle with sides the vector $\vec{w}$ and its horizontal component $\vec{w_x}$. Let the directional angle be α. Then

Let r be the $|w|$.

$\cos\alpha = |w_x|/r = |w_x|/|w|$

$\sin\alpha = |w_y|/r = |w_y|/|w|$.



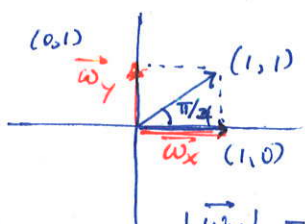
$|\vec{w_x}| = r\cos\alpha$

$|\vec{w_y}| = r\sin\alpha$.

Finding horizontal and vertical components

Example

Consider the vector $\vec{w} = \langle 1, 1 \rangle$, with direction angle $\alpha = \pi/4$. Find the horizontal and vertical components.



$\vec{w_x} = \langle 1, 0 \rangle$

$\vec{w_y} = \langle 0, 1 \rangle$.

$|\vec{w_y}| = r\sin\alpha = r\sqrt{2}/2$

$|\vec{w_x}| = r\cos\alpha = r\frac{\sqrt{2}}{2}$

$r = |w| = \sqrt{(1-0)^2+(1-0)^2} = \sqrt{2}$.

$|\vec{w_x}| = 1$

$|w_y| = 1$.

Finding the component form given magnitude and direction

Example

Find the vector $\vec{w} = \langle a, b \rangle$, with direction angle $\alpha = 330°$ and magnitude 40.

$a = r\cos\alpha$

$b = r\sin\alpha$.

$r = 40$

recall

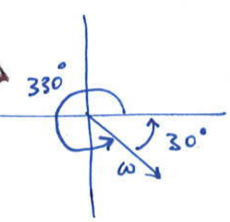
$\Rightarrow w = (\pm|w_x|, \pm|w_y|)$ with >0, <0

$w = \left(40\frac{\sqrt{3}}{2}, -\frac{1}{2}\cdot 40\right)$

$\sin\alpha = -\sin 30° = -1/2$

$\cos\alpha = \cos 30° = \frac{\sqrt{3}}{2}$

4th quadr.



$\sin\alpha < 0$

$\cos\alpha > 0$.

$|\sin 330°| = |\sin 30°|$.

$|\cos 330°| = |\cos 30°|$.

Operations on vectors - Algebraic

Theorem

Let $\vec{A} = \langle a_1, a_2 \rangle$ and $\vec{B} = \langle b_1, b_2 \rangle$ and k is a scalar (i.e. a real number), then

1. $k\vec{A} = \langle ka_1, ka_2 \rangle$ *(scalar product)*
2. $\vec{A} + \vec{B} = \langle a_1 + b_1, a_2 + b_2 \rangle$
3. $\vec{A} - \vec{B} = \langle a_1 - b_1, a_2 - b_2 \rangle$
4. $A.B = a_1b_1 + a_2b_2$ ***(dot product)***

Remark

Note that the dot product of two vectors is a number and the scalar product is a vector.

Operations on vectors - Algebraic

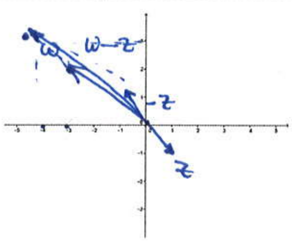
Example

Let $\vec{w} = \langle -3, 2 \rangle$ and $\vec{z} = \langle 1, -1 \rangle$. Find the following (both algebraically and geometrically - where possible)

$\vec{w} - \vec{z} =$ $\langle -3, 2 \rangle - \langle 1, -1 \rangle = \langle -3-1, 2+1 \rangle = \langle -4, 3 \rangle$

$2\vec{w} + 3\vec{z} =$ ...

$\vec{w}.\vec{z} =$ $(-3).1 + 2.(-1) = -3-2 = -5$



Angle between two vectors - application of dot product

Theorem (Angle between two vectors)

If $\vec{A}$ and $\vec{B}$ are two non-zero vectors and α is the angle between them, then

$$\cos \alpha = \frac{\vec{A}.\vec{B}}{|A||B|}.$$

Proof.

Apply the law of cosines to the triangle with sides a, b and c, where $a = |\vec{A}|$, $b = |\vec{B}|$ and $c = |\vec{A} - \vec{B}|$. □

$\vec{A} = \langle a_1, a_2 \rangle$

$\vec{B} = \langle b_1, b_2 \rangle$

$\vec{A} - \vec{B} = \langle a_1 - b_1, a_2 - b_2 \rangle$.

$|A-B|^2 = |A|^2 - |B|^2 - 2|A||B| \cos \alpha \Rightarrow$

(see back)

Angle between two vectors - application of dot product

Example

Find the smallest possible angle between each two pairs $\langle -5, 9 \rangle$ and $\langle 9, 5 \rangle$.

$\langle -5, 9 \rangle . \langle 9, 5 \rangle = -5.9 + 9.5 =$

$= -45 + 45 = 0.$

$\cos \alpha = \frac{A.B}{|A||B|} = 0.$

$\Rightarrow \alpha = \pi/2.$

Angle between two vectors - application of dot product

Remark

If $\vec{A}$ and $\vec{B}$ are two non-zero vectors and α is the angle between them and if $\vec{A}.\vec{B} = 0$, then

$$\cos \alpha = \frac{\vec{A}.\vec{B}}{|A||B|} = 0.$$

Thus $\alpha = 90°$. That is, the vectors $\vec{A}$ and $\vec{B}$ are **perpendicular**.

Remark

If $\vec{A}$ and $\vec{B}$ are two non-zero vectors and α is the angle between them and if $\cos \alpha = \pm 1$, then $\alpha = 0°$ or $\alpha = 180°$. That is, the vectors $\vec{A}$ and $\vec{B}$ are **parallel**.

Unit Vectors

Definition

Vectors with length 1 are called **unit vectors**.

Two such vectors are $\vec{i} = \langle 1, 0 \rangle$ and $\vec{j} = \langle 0, 1 \rangle$.

Remark (Important Observation)

We can write every vector $\langle a_1, a_2 \rangle$ in the following way, called a **linear combination**:

$$\langle a_1, a_2 \rangle = a_1 \langle 1, 0 \rangle + a_2 \langle 0, 1 \rangle = a_1\vec{i} + a_2\vec{j}.$$

Unit Vectors

Example

Write the vector $\langle 2, -6\rangle$ as a linear combination of the unit vectors $\vec{i}$ and $\vec{j}$.

$$\langle 2,-6\rangle = 2\langle 1,0\rangle + (-6)\langle 0,1\rangle$$
$$= 2\vec{i} + (-6)\vec{j}$$

Opinion poll :-)

Question

Your opinion on using slides instead of the classical lecture.

(A) *I think using slides has improved the quality of the lecture.*

(B) *I like slides more - at least I can stay awake.*

(C) *Slides are even worse than what we have been doing until now! I like the regular lecture better.*

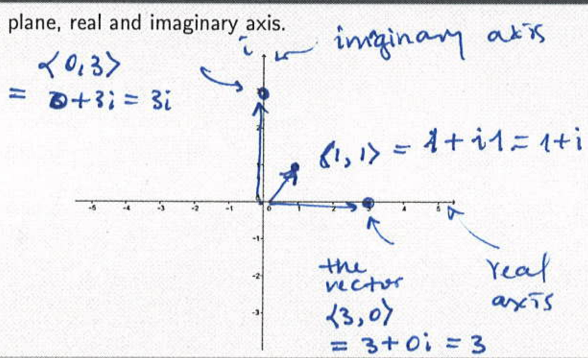
(D) *I am definitely a fan of the regular lecture format.*

(E) *I hate both... so much.*

Complex Numbers - Chapter 7.4

Definition

Complex plane, real and imaginary axis.



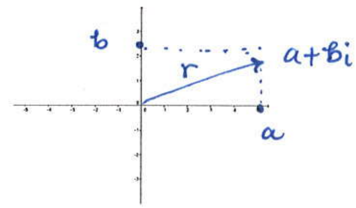
Complex Numbers - absolute value

Definition (Absolute value of a complex number)

The absolute value of the complex number $a + ib$ is defined by

$$r = |a + ib| = \sqrt{a^2 + b^2}.$$

Note that this is the distance from the center of the complex plane to the point (a, b).

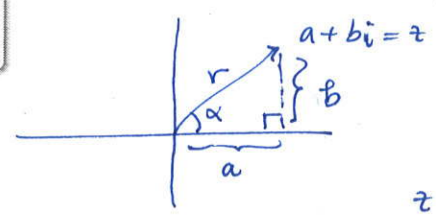


Complex Numbers - trig form

Definition (Trigonometric form of complex numbers)

Consider the complex number $z = a + ib$. Let $r = |a + ib| = \sqrt{a^2 + b^2}$ and let α be the angle between $\langle a, b\rangle$ and the positive x-axis. Then the trigonometric form of the complex number z is

$$z = r(\cos\alpha + i\sin\alpha).$$



$\cos\alpha = a/r$

$\sin\alpha = b/r$

$z = a+ib = r\cos\alpha + i\,r\sin\alpha$

Complex Numbers - trig form

Example (Write the complex number in trig form)

Write the complex number $-2\sqrt{3} + 2i$. in trig form.

$a = \cos\alpha \cdot r = -2\sqrt{3}$

$b = \sin\alpha \cdot r = 2.$

$r = \sqrt{a^2+b^2} = \sqrt{(-2\sqrt{3})^2 + 2^2} = 4.$

$\left.\begin{array}{l}\cos\alpha = -\sqrt{3}/2 \\ \sin\alpha = 1/2\end{array}\right\}\ \alpha = \frac{5\pi}{6}\ (= 150^\circ)$

$z = 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right).$

Complex Numbers - trig form

Example (Write the complex number in standard form)

Write the complex number $\sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$ in the form $a + ib$.

$(r = \sqrt{2})$

$\cos(\pi/4) = \sqrt{2}/2$

$\sin(\pi/4) = \sqrt{2}/2$

$\Rightarrow z = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$

$= \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1 + i$

Complex Numbers - trig form

Theorem

Let $z_1 = r_1(\cos\alpha_1 + i\sin\alpha_1)$ and $z_2 = r_2(\cos\alpha_2 + i\sin\alpha_2)$, then

$$z_1 z_2 = r_1 r_2(\cos(\alpha_1 + \alpha_2) + i\sin(\alpha_1 + \alpha_2))$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\alpha_1 - \alpha_2) + i\sin(\alpha_1 - \alpha_2))$$

Proof.

Just try to compute $z_1 z_2$ and $\frac{z_1}{z_2}$. □

on the back? $z_1 z_2$ – just multp.

$\frac{z_1}{z_2}$ – multip by conjug. of denomin.

Complex Numbers - trig form

Example (Product in trig form)

Use trigonometric form to find $z_1 z_2$, if $z_1 = -2 + 2i\sqrt{3}$ and $z_2 = \sqrt{3} + i$.

first convert z_1 & z_2 to trig form.

for z_1: $r_1 = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4.$

$\cos\alpha_1 = a_1/r_1 = -1/2$

$\sin\alpha_1 = b_1/r_1 = \sqrt{3}/2$

$\Rightarrow \alpha_1 = 120°$

$z_1 = 4(\cos 120° + i\sin 120°)$

for z_2: $r_2 = \sqrt{(\sqrt{3})^2 + (1)^2} = \sqrt{3+1} = 2$

$\cos\alpha_2 = a_2/r_2 = \sqrt{3}/2$

$\sin\alpha_2 = b_2/r_2 = 1/2$

$\Rightarrow \alpha_2 = 30°$

$z_2 = 2(\cos 30° + i\sin 30°)$

Example cont:

$z_1 \cdot z_2 =$

$= 2 \cdot 4\,(\cos(120° + 30°) + i\sin(120° + 30°))$

$= 6(\underbrace{\cos 150°}_{-\frac{\sqrt{3}}{2}} + i\underbrace{\sin 150°}_{\frac{1}{2}}).$

$= -4\sqrt{3} + 4i$