Prereq. Ch. 1 & 2

Numbers

Consider the following sets (collect. of elements).

$\{0, 1, 2, \ldots\}$ - natural #s $\mathbb{N}$

$\{\ldots -1, 0, 1, 2, \ldots\}$. integrs $\mathbb{Z}$

$\{\ldots \frac{1}{2}, 1, \frac{5}{3}, \frac{7}{8}, \ldots\}$. rationals $\mathbb{Q}$

$\{\sqrt{2}, \pi, e, \sqrt{7}, \ldots\}$. reals $\mathbb{R}$.

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

$$\mathbb{R} \setminus \mathbb{Q} \subset$$

rational numbers : $\left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \right\}$. b/c are a ratio

ex: $\frac{3}{4}$, $\frac{4}{2} = 2$, 1 b/c $1 = \frac{1}{1}$

reals = rational $\cup$ irrational

irrational are the ones that have nonrep. decimals

eg. $1/6 = 1,666\ldots$

$\pi = 3,142857\ldots$

cannot be written as a ratio

Properties of the real #s (very natural). $a, b \in \mathbb{R}$ (opt)

- $a+b \in \mathbb{R}$ eg. $2+\sqrt{2} \in \mathbb{R}$ closure
  $ab \in \mathbb{R}$ $2.\sqrt{2} \in \mathbb{R}$.

- $a+b = b+a$ & $ab = ba$ commutat.

- $(a+b)+c = a+(b+c)$ assoc.
  $(ab)c = a(bc)$

- $a(b+c) = ab+ac$ distrib.

- $0+a = a$ & $1.a = a$ identity

- $a+(-a) = 0$ multip & add inverse.
  $a.\left(\frac{1}{a}\right) = 1$

Q: Does every real # have a multip inverse? (all exc. 0).

Properties of $>, <, =$ $a>b$ or $b>a$ or $a=b$ (opt).

- if $a>b \Rightarrow b<a$

- $\cancel{\text{if}}$ $a=a$

- if $a=b$ & $b=c \Rightarrow a=c$

- $a=b \Rightarrow b=a$.

Abs. value:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0. \end{cases}$$

not as confusing as you think!

example

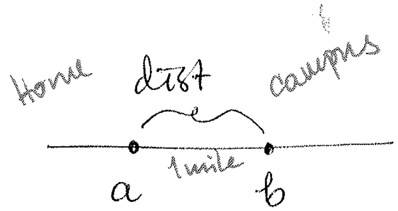
- $a = 2 \Rightarrow |a| = |2| = 2$, because $2 \geq 0$.
- $a = -5 \Rightarrow |a| = |-5| = -(-5)$, because $5 < 0$
  $= \underline{\underline{5}}$ (!)

(**Note:** $|a| \geq 0$ always!)

Properties of $|a|$:

- $|-a| = |a|$;
- $|a| \geq 0$, $\forall a$
- $|ab| = |a||b|$ eg. $|3.(-5)| = |3||-5| = 15$.
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ eg. $\left|\frac{3}{-5}\right| = \frac{|3|}{|-5|} = \frac{3}{5}$

Applicat.: Dist. betw. two points on the # line



$\text{dist} = |a-b| = |b-a| = b-a$

(if $b \geq a$)

eg $a=1$ $b=2$ $\Rightarrow$ dist $= 1$ no matter if you go $a \to b$ or $b \to a$.

Exponential expr.

$$a^n = \underbrace{a \cdot a \cdot \ldots \, a}_{n}$$

eg $6^3 = 6 \cdot 6 \cdot 6$.

Order of operat: 1) parenthesis 2) exp, abs val. → multip, div. → add, subtr.

Ex: evaluate:

$$5 - 2(\underbrace{3 - 4 \cdot 2}_{\text{do this first.}})^2 = 5 - 2\underbrace{(3 - 8)}_{\text{parenth.}}^2 =$$

$$= 5 - 2\underbrace{(-5)^2}_{\text{exp.}} = 5 - \underbrace{2 \cdot 25}_{\text{multip.}} = 5 - 50 = -45$$

simplify.

$$-4x - \underbrace{(6 - 7x)}_{\text{open parenth.}} = -4x - 6 + 7x = -6 + \underbrace{7x - 4x}_{\text{group.}}$$

$$= -6 + 3x$$

$$-(6 - 7x) = (-1)(6 + (-7x)) = (-1)(6) + (-1)(-7x).$$

distrib. prop.

2). More exponentials expr.

→ negative exponents: $\boxed{a^{-n} = 1/a^n}$ $= \left(\frac{1}{a}\right)^n$ **Q1** eg. $5^{-3} = \frac{1}{5^3}$

$$\Rightarrow \boxed{\left(\frac{a}{b}\right)^{-m}} = \left(\frac{1}{\left(\frac{a}{b}\right)}\right)^m = \boxed{\left(\frac{b}{a}\right)^m}$$

$$\parallel$$

$$A^{-m} = \frac{1}{A^m}$$

alternatively: (better).

$a^{-1} = 1/a$

$a^{-n} = a^{(-1).n} = (a^n)^{-1} = 1/a^n$

$= (a^{-1})^n = \left(\frac{1}{a}\right)^n = \left(\frac{1}{a}\right)^n$

Note $a^0 = 1$.

1)
Properties:

- $a^m a^n = a^{m+n}$
- $\frac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{m.n}$
- $(ab)^m = a^m b^m$
- $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}.$

these work when
$a \in \mathbb{R}$
& $m, n \in \mathbb{R}$! (pa.3).
(not only $\mathbb{Z}$).

Examples: Simplify (opt)

P2/56

$$\frac{6a^{9s}b^{4t}}{-9a^{3s}b^{8t}} =$$

$$= \frac{\cancel{6}^{2}}{\cancel{-9}_{3}} \cdot \frac{a^{9s}}{a^{3s}} \cdot \frac{b^{4t}}{b^{8t}} =$$

$$= -\frac{2}{3} \cdot a^{9s-3s} \cdot b^{4t-8t}$$

$$= -\frac{2}{3} \cdot a^{6s} \cdot b^{-4t}$$

P2/52

exp > multip.

$$(a^2)^{m+2} \cdot (a^3)^{4m} = a^{2(m+2)}\, a^{3\cdot 4m}$$

$$= a^{2m+4} \cdot a^{12m} \underset{\text{for exp.}}{\overset{\text{prod. rule}}{=}} a^{(2m+4)+(12m)} =$$

$$= a^{14m+4}$$

P2/46

$$\left(\frac{-1}{2a}\right)^{-2} = \left(\frac{2a}{-1}\right)^2 = \frac{(2a)^2}{(-1)^2} = \frac{2^2a^2}{1} = 4a^2$$

$-2 \Rightarrow 2.$
pos. exp.

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$\left(\left(\frac{a}{b}\right)^{-1}\right)^2 = \left(\frac{b}{a}\right)^2$$

use A vs a.

$$\left(\frac{-1}{2a}\right)^{-2} = \left(\left(\frac{-1}{2a}\right)^{-1}\right)^2 = \left(\frac{2a}{-1}\right)^2.$$

Rational/Real exponents (P3).

- if $a^2 = b \Rightarrow$ a is square root of b
  (eg. $a^2 = 3 \Rightarrow a = \sqrt{3}$)
- if $a^3 = b \Rightarrow$ a is cube root of b
  (eg. $a^3 = 5 \Rightarrow a = \sqrt[3]{5}$).
- if $a^n = b \Rightarrow$ a is n^{th} root of b.

Note: $0^{1/n} = 0$.

if $n \in \mathbb{N}$ (even) $a \geq 0 \Rightarrow a^{1/n}$ - the positive (principal) n^{th} root of a

if $n \in \mathbb{N}$ (odd.) $a \in \mathbb{R} \Rightarrow a^{1/n}$ - the n^{th} root.

note when n is even & $a < 0 \Rightarrow a^{1/n} \notin \mathbb{R}$.
eg. $a = -1$, $n = 2 \Rightarrow \sqrt{-1} \notin \mathbb{R}$ but $\mathbb{C}$!

- $\boxed{a^{m/n} = (a^{1/n})^m}$ same as before $a^{AB} = (a^A)^B$ but here $A = 1/n$, $B = m$

Notation. $a^{1/n} = \sqrt[n]{a}$ & $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

- $\sqrt[n]{ab} = (ab)^{1/n} = a^{1/n} b^{1/n} = \sqrt[n]{a}\sqrt[n]{b}$
- $\sqrt[n]{\frac{a}{b}} = \left(\frac{a}{b}\right)^{1/n} = \frac{a^{1/n}}{b^{1/n}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$
- $\sqrt[m]{\sqrt[n]{a}} = (a^{1/n})^{1/m} = a^{1/m \cdot n} = \sqrt[m.n]{a}$.

$a^{1/n \cdot 1/m}$

Examples:

- $\sqrt[3]{y} \cdot \sqrt[4]{2y} = y^{1/3} \cdot (2y)^{1/4} = y^{1/3} \cdot 2^{1/4} \cdot y^{1/4}$

$= \sqrt[4]{2}\,(\underbrace{y^{1/3} \cdot y^{1/4}}_{y^{1/3+1/4}}) = \sqrt[4]{2}\ \underbrace{y^{7/12}}_{\sqrt[12]{y^7}} \stackrel{\triangle}{=} \boxed{\underbrace{2^{1/4}}_{\substack{\shortparallel \\ 2^{3/12}}}} \cdot y^{7/12} = \sqrt[12]{2^3 y^7}$

$= 2^{3/m} \cdot y^{7/m} = (2^3 \cdot y^7)^{1/m}$ $m = 12$

b/c $\frac{1}{4} = \frac{3}{12}$.

$= \sqrt[12]{2^3 y^7}$

(opt)

- $\left(\frac{a^{3/2} \cdot b^{2/3}}{a^2}\right)^3 = \frac{a^{(3/2) \cdot 3} \cdot b^{(2/3) \cdot 3}}{a^{2 \cdot 3}} = \frac{a^{9/2} \cdot b^2}{a^6} = a^{9/2 - 6} \cdot b^2$

$= a^{-3/2} \cdot b^2 =$

$= \frac{b^2}{a^{3/2}} = \frac{b^2}{\sqrt[3]{a^2}}$.

(P4) Polynomials

Def: $n \in \mathbb{N}$, $a_0, \ldots, a_n \in \mathbb{R}$

leading coeff. degree constant term

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

is a polynomial in one variable x.

eg. $3x^5 + 7x + 1 = P(x)$

$3 = P(x)$ ← constant polynomial.

$3x^8 = P(x)$ ← monomial ie. $a_k x^k$

when $n = 2$ - quadratic

$n = 3$ cubic

- Evaluation a polynomial at a number. (think of the polynomial as a function) (ie. "plug in").

eg. $P(x) = 3x^3 + 8$

$P(2) = 3.(2)^3 + 8 = 3.8 + 8 = 32.$

- Adding / subtr. polynomials: (use distributive law & group).

eg $P(x) = 3x^3 + 8$

$Q(x) = 4x^4 + x^3 - x^2 + x$

$$P(x) + Q(x) = \overbrace{3x^3 + 8 + 4x^4 + x^3}^{(3+1)x^3} + x^2 + x$$

$$= 4x^4 + 4x^3 + x^3 + x + 8$$

- Multiplying poly:

$$(a+b)(\underbrace{c+d}_{A}) = (a+b)A = aA + bA = a(c+d) + b(c+d)$$

$\rightarrow$ use distributivity $= ac + ad + bc + bd.$

eg: $(x^2+3x+4)(2x-3) = 2x(x^2+3x+4) - 3(x^2+3x+4).$

$$2x(x^2) + 2x(3x) + 2x(4) - 3(x^2) - 3(3x) - 3(4).$$

alternatively $= x^2(2x-3) + 3x(2x-3) + 4(2x-3).$

$\rightarrow$ open parenthesis.

Note
$$(a+b)^2 = a^2+b^2+2ab$$
$$(a-b)^2 = a^2+b^2-2ab$$
$$(a+b)(a-b) = a^2+b^2.$$

- Application: 1) multiplying real numbers:

$$(3+\sqrt{6})(3-\sqrt{6}) = 3^2-(\sqrt{6})^2 = 9-6 = 3.$$

2) rationalite denominator: (simplify).

$$\frac{3}{3-\sqrt{6}} = \frac{3(3+\sqrt{6})}{(3-\sqrt{6})(3+\sqrt{6})} = \frac{9-3\sqrt{6}}{3} = 3-\sqrt{6}.$$

Polynomial division:

$$\text{Polynomial } \underbrace{P(x)}_{\text{divident}} = \underbrace{Q(x)}_{\text{quotient}}\,\underbrace{D(x)}_{\text{divisor}} + \underbrace{R(x)}_{\text{reminder}}$$

$\deg R(x) < \deg D(x)$
($R(x)$ can be 0).

Example:

$$\begin{array}{r|l} \overbrace{x^2-3x-9}^{P(x)} & \overbrace{x-5}^{D(x)} \\ -\;x^2-5x & \underbrace{x+2}_{Q(x)} \\ \hline 2x-9 & \\ -\;2x-10 & \\ \hline \underbrace{1}_{R(x)} & \end{array}$$

$\Rightarrow \quad x^2-3x-9 = (x-5)(x+2)+1.$

(opt)
(applicat: ⇝ as a rational expression

$$\frac{x^2-3x-9}{x-5} = x+2+\frac{1}{x-5}$$)

<u>Factoring polynomials</u>: – "the opposite of multiplication".
(find two (or more) polynomials that you multiply to get the original polynomial). Q2
we call these <u>factors</u>!

- greatest common factor $3x^2+3x = 3x(x+1)$.
- group. $\underbrace{x^3+x^2}+\underbrace{4x+4} = x^2(x+1)+4\cdot(x+1) = (x+1)(x^2+4$
- factor square polynomials* $\begin{cases} ax^2+bx+c & a=1. \\ ax^2+bx+c & a\neq 1. \end{cases}$
- special products (opt)
- trial & error.
- substitution $x^4+2x^2+1 = a^2+2a+1 = (a+1)^2 = (x^2+1)^2$.
  $\underline{x^2=a}$.
- use division algoritm.

<u>eg</u>. x^4+2x^2+1 & know that x^2+1 is a facto

$\Rightarrow x^4+2x^2+1$ divided by x^2+1

$= x^2+1$ & reminder 0

$\Rightarrow (x^2+1)(x^2+1) = x^4+2x^2+1$. $(x-x_0)(x-x_1)$.

$\circledast$ <u>Ex</u>: $(x-7)(x+2) = x^2-7x+2x-14. = x^2-5x-14 = ax^2+bx+c$

$\overset{\shortparallel}{(-7).2}$ $\quad a=1$ $\quad c=-7.2! = x_1 \cdot x_0$.

Ex: $a \neq 1$

$$ax^2+bx+c = a(x-x_0)(x-x_1).$$

$$\Rightarrow x_1 . x_0 . a = c$$

Note: Sometimes we cannot write a polynomial as a product of 2 other eg. x^2+x+1.
(we call these irreducible)

Q3

P6 (Rational Expressions).

Definition: A rational expression: ratio of two polynomials, in which the denominator isn't 0.

Def: Domain of a rational expression:
the set of all real numbers for which when we evaluate the denomin. at them we don't get 0.

eg. $\frac{1}{2-x}$, domain: { all $\mathbb{R}$ except 2 }.

- Addition, subtract., multipl & division → like for rational numbers!

Ex $$3\left(\frac{x^2+5x+1}{x-3}\right) \div \left(\frac{x-2}{x-3}\right) + \frac{1}{x-1}$$

$$3\left(\frac{x^2+5x+1}{\cancel{x-3}} \cdot \frac{\cancel{x-3}}{x-2}\right) + \frac{1}{x-1}$$

$$3\left(\frac{x^2+5x+1}{x-2}\right) + \frac{1}{x-1} = \underbrace{\frac{\overset{x-1}{3(x^2+5x+1)}}{x-2} + \frac{\overset{x-2}{1}}{x-1}}_{(x-2)(x-1) \leftarrow \text{GCD}}$$

$$= \frac{3(x-1)(x^2+5x+1)+x-2}{(x-2)(x-1)}.$$

Complex fractions (has rational expressions in both numerator & denominator).

$$\frac{\frac{1}{2a^{-1}} - \frac{1}{4}}{\underbrace{\frac{3}{a}^{6} + \frac{a}{6}^{a}}_{6a}} = \frac{\frac{1}{2\frac{1}{a}} - \frac{1}{4}}{\frac{18}{6a} + \frac{a^2}{6a}} = \frac{\frac{1}{\frac{2}{a}} - \frac{1}{4}}{\frac{18+a^2}{6a}} = \frac{\frac{a}{2} - \frac{1}{4}}{\frac{18+a^2}{6a}}$$

$= \frac{a}{2}$

simplify numerat. & denominator

$$= \frac{\frac{2a-1}{4}}{\frac{18+a^2}{6a}} = \frac{2a-1}{4} \div \frac{18+a^2}{6a} = \frac{2a-1}{4} \cdot \frac{6a}{18+a^2}$$

write as division of fractions

$$= \frac{6a(2a-1)}{4(18+a^2)}$$ = open paranthesis!

P7 (complex numbers).

Q1: Recall $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, where do $\mathbb{C}$ stay?

Q2: ~~What's the multip. inverse of i?~~

Recall that sometimes we can't write a polynomial as product of non-const poly's with coeff. $\in \mathbb{R}$, ~~but~~

eg. x^2+1

but we can "over the complex mubers"

$$x^2+1 = (x-i)(x-i), \quad \sqrt{-1} = i$$

Def: complex number: (in strandard form).

$$\underbrace{a}_{a=\text{real part}} + \underbrace{b}_{b=\text{imaginary part}} i \qquad i^2 = -1, \quad a, b \in \mathbb{R}$$

Adding / Subtr. / Multip... - like polynomials

eg: $\underbrace{(3+2i)(1-i)} + (5-2i) \overset{\text{distribute}}{=}$

$= 3.1 - 3.i + 2i.1 - 2\underbrace{i^2}_{2} + 5 - 2i \overset{\text{eval. } i^2}{=}$

$= 3 - 3i + 2i - 2 + 5 - 2i \overset{\text{group}}{=}$

$= \underline{\underline{6 - 3i}}.$

Note $(a+ib)(a-ib) = a^2+b^2 \in \mathbb{R}$!

these are complex conjugates of each other.

Q5 -

Dividing complex numbers:

(use the complex conjugates)

Ex: divide $8-i$ by $2+i$

$\Rightarrow \dfrac{8-i}{2+i} = ? \Rightarrow \dfrac{8-i}{2+i} \cdot \dfrac{(2-i)}{(2-i)} = \dfrac{15-10i}{5} = \underline{\underline{3-2i}}$

Roots of neg. numbers:

$\sqrt{-n}$, where $n \in \mathbb{R}_{\geq 0}$

$\sqrt{-n} = i\sqrt{n}$

eg. $\sqrt{-50} = i\sqrt{50}$