Chapter 1

(1.1).

Def: An equation: a statement that two algebraic expressions are equal

eg. $2x = 3 + y$.

Def: A linear equation in one variable: equation of the form $ax + b = 0$, $a \neq 0$, $a, b \in \mathbb{R}$.

Def: Solution to a (linear) eq. all x-s for which the equation is true:

eg. $2x + 3 = 0 \Rightarrow x = -3/2$. has 1 solution.

eg. $2x + 3 = 2x - 1 \Rightarrow 0 \cdot x = -4$ no solutions. (inconsitent)

eg. $2x + 3 = 2x + 3 \Rightarrow 0 \cdot x = 0$ all $\mathbb{R}$ are solutions (we call this an identity).

Properties of equalities: If $A = B$ then:

$A \pm C = B \pm C$

$AC = BC$, (for $C \neq 0$) - not even needed.

$\frac{A}{C} = \frac{B}{C}$, (for $C \neq 0$)

eg. if $3x = 2 \Leftrightarrow \frac{3x}{5} = \frac{2}{5}$

Ex: $5(3x-2) = 5 - 7(x-1)$.

$15x - 10 = 5 - 7x + 7$ (+7x on both sides).

$\underbrace{15x+7x}_{22x} - 10 = 5 \underbrace{- 7x + 7x}_{0} + 7$

$22x - 10 = 5 + 7 = 12$ (+10 on both sides).

$22x - 10 + 10 = 12 + 10$

$22x = 22$ (divide by 22 on both sides).

$\underline{x = 1}$.

Ex: $\frac{y}{y-3} + 3 = \frac{3}{y-3}$ Find the domain first!

⚠ when we see a rational expression, always the first thing we do is find the domain.

Domain: $y \neq 3$

$\frac{y}{y-3} - \frac{3}{y-3} = -3$ $\boxed{\text{or}}$ $(y-3)\left(\frac{y}{y-3} + 3\right) = (y-3)\left(\frac{3}{y-3}\right)$

$y + 3y - 9 = 3 \Rightarrow y = 3$

$\frac{y-3}{y-3} = -3$ if $y \neq 3 \Rightarrow$ cancel $\Rightarrow 1 = -3 \Rightarrow$ no solut.

Ex: $|x-5| = 2$ recall: $|a| = \begin{cases} a & \text{if } a > 0 \\ -a & \text{if } a < 0 \end{cases}$.

$x - 5 = 2$ if $x - 5 > 0$ ie. if $x > 5$ or $x - 5 = -2$ if $x - 5 < 0$. ie. if $x < 5$

$x = 7$ $(x > 5)$ ✓ $x = 3$ $(x < 5)$ ✓

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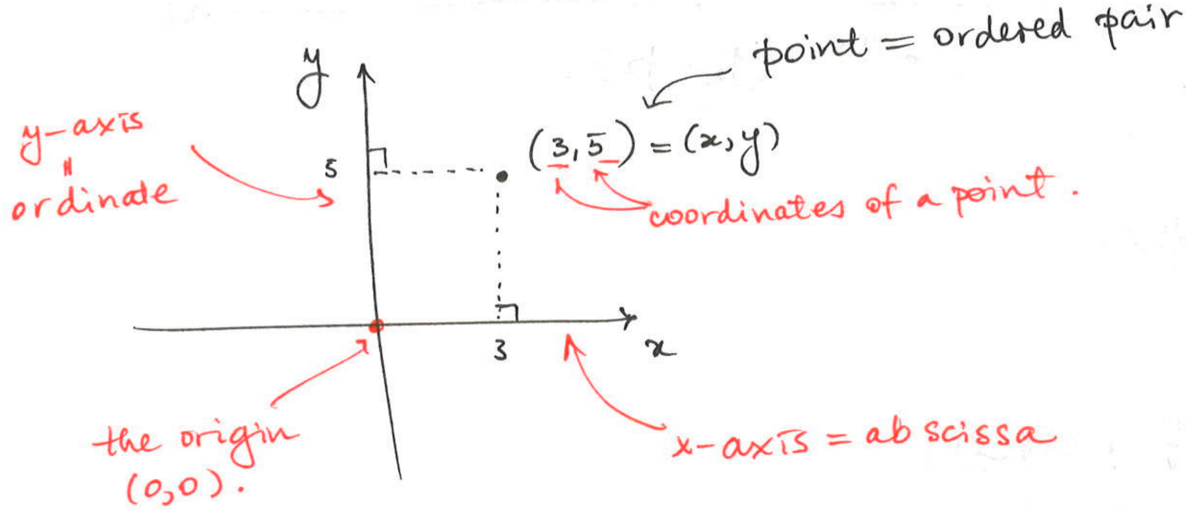
Cartesian coordinate system (= rectangular coord. sys).

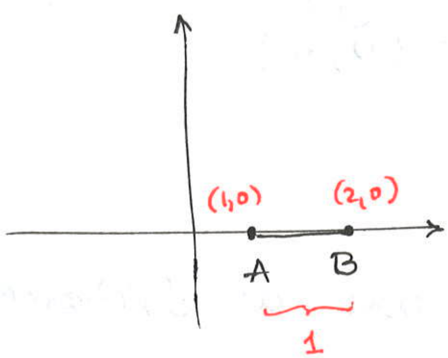
(b/c the x & y are ⊥ & the lines to them that gives us the coordinates are also ⊥ to the axis).

the catesian plane = xy plane



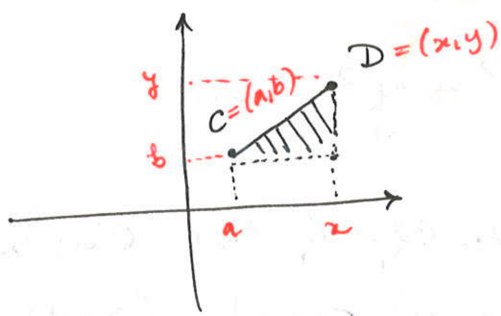
graphing a point - finding where it lies on the xy-plane.

Distance formula



(Intuit. from the line-case)
i.e. A, B lie on a line

$|A - B|$

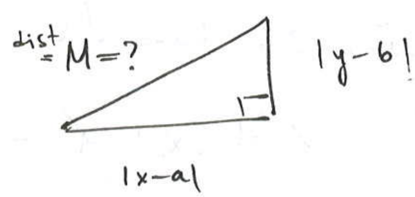


Let M be the distance between C & D

dist. b/w a & x = $|x-a|$

dist. b/w y & b = $|y-b|$

look at the Δ

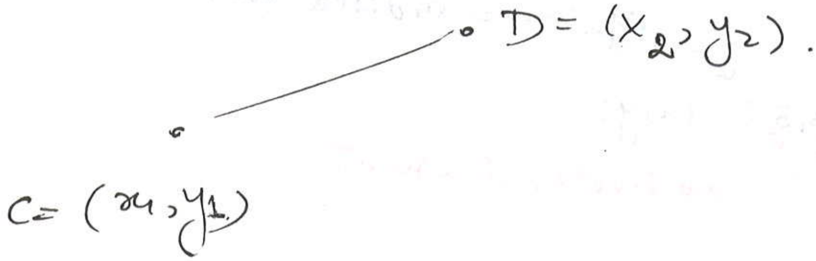


$M^2 = \; = |x-a|^2 + |y-b|^2$ ($a^2+b^2=c^2$ pythag. thm).

$$\boxed{M = \sqrt{|x-a|^2 + |y-b|^2} = \left(\text{since } \begin{matrix} x>a \\ y>b \end{matrix}\right) = \sqrt{(x-a)^2+(y-b)^2}}$$

Similary we can obtain a midpt. f-a for the dist b/w C & D. if $C = (x_1, y_1)$
$D = (x_2, y_2)$

$$\Rightarrow \boxed{\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \text{midpt. of the line seg.}}$$



Equation of a circle

Circle with center (h, k) & radius r $(r > 0)$. is

$$\boxed{(x-h)^2 + (y-k)^2 = r^2}$$ standard form of circle eq.

If the circle is centered at the origin $\Rightarrow h = 0, k = 0$
$= (0, 0)$

$\Rightarrow$ eq. is $x^2 + y^2 = r^2$.

Q: Why is this the eq. of the circle?

A: B/c these are all points (x, y) which are at distance r from the point (h, k). (Recall the dist. formula).

Q: What do we do if the circle is given by this eq: $x^2 + 6x + y^2 - 5y = -\frac{1}{4}$

what are the radius & center.

A: write it as: $(x-h)^2 + (y-k)^2 = r^2$.

Ex: $x^2+6x+y^2-5y=-1/4.$

$$\underbrace{x^2+6x+9}+\underbrace{y^2-5y+\frac{25}{4}}=-\frac{1}{4}+9+\frac{25}{4}$$

$$(x+3)^2+\left(y-\frac{5}{2}\right)^2=15$$

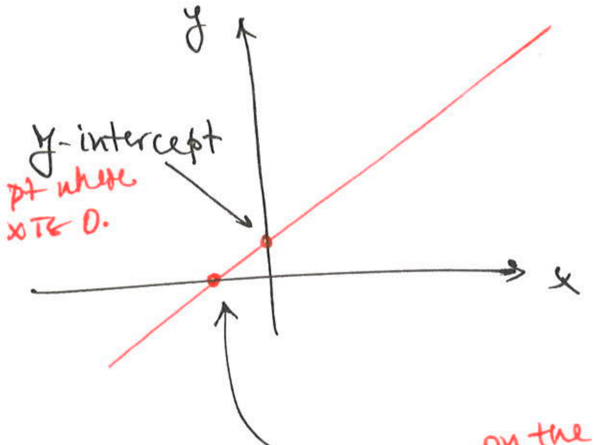
$\Rightarrow$ rad $=\sqrt{15}$ & center $\left(-3,+\frac{5}{2}\right)$!

Equat. of a line:

$\boxed{Ax+By=C}$ for $A, B, C \in \mathbb{R}$. - standard form of line eq.

(this is also a linear eq. in two variables).

x-intercept & y-intercept.

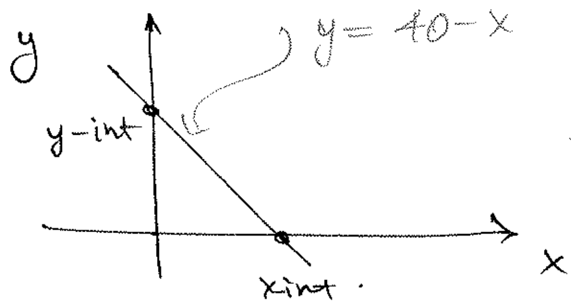


x-intercept = pt on the line where y = 0.

that is plug in the eq. of the line y=0 to find the x intercept!

(b/c this point satisfies the eq. of the line)

Ex: $y = 40 - x$



for x_intercept $y=0 \Rightarrow 0 = 40 - x \Rightarrow x = 40$

for y- intercept $x=0 \Rightarrow y = 40 - 0 = 40$

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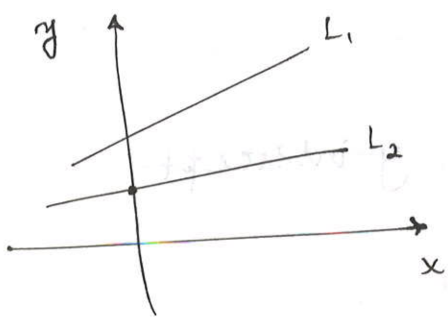
. Linear equation in two variables: Equation that looks like

$$Ax + By + C = 0. \quad \text{(equations of lines).}$$

. Example: Consider the following two lines:



Q: How do we distinguish them? (what properties)

A: We can by intercepts points lying on them or something that includes in a sense all that - the slope.

Def: The slope of a line that passes through (x_1, y_1) & (x_2, y_2) whith $x_1 \neq x_2$ is

$$(y_2 - y_1)/(x_2 - x_1) = m.$$

The slope measures the "steepness" of the line.

[note: when $x_1 = x_2$ the line is vertical - no slope! (or ∞)].

Def: The equation of a line (point-slope form) is:

$$\boxed{y - y_1 = m(x - x_1)} \Leftrightarrow y - mx = -mx_1 + y_1.$$

$$\Leftrightarrow y = mx + \underbrace{mx_1 - y_1}_{b}$$

this is the y-intercept

$$\boxed{y = mx + b}$$

(slope-intercept form)

all we need is the slope m & a point through which the line is going (x_1, y_1).

Note: This is the eq. of the line because the slope of the line remains the same no matter which two pts on the line we pick.

that is $\dfrac{y-y_1}{x-x_1} = m$, for $\forall$ (x,y) on the line.

<u>Ex</u>: • Find the slope of the line through the pts $(-3,7)$ & $(5,7)$

$$\Rightarrow m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-7}{5+3} = 0. \quad \text{(horizontal line).}$$

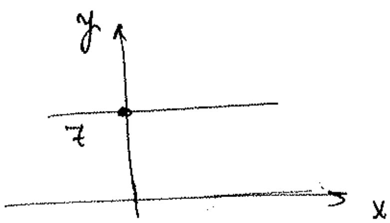
• Find the eq. of the line

$y = mx + b$, b is the y-intercept.

plug in one of the pts & m

$7 = 0(-3) + b \Rightarrow b = 7$

$y = mx + b \Rightarrow y = 0 \cdot x + 7 \Rightarrow y = 7$



<u>Prop</u>: • The slope of parallel lines is the same!
(non-vertical?)

• The slopes of two perpendicular lines are also related!

if m_1 & m_2 are the slopes then $m_1 \cdot m_2 = -1$.

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Def: Quadratic equation in one variable

$$ax^2+bx+c=0, \quad \text{with } a,b,c\in\mathbb{R}$$
$a\neq 0$! (otherwise it's linear).

Solving quadratic equations (ie finding solutions)

Method 1: By factoring:

Ex: Let $(x-3)(x+2)=0$ then the solutions are $x=3$, $x=-2$, b/c when we plug them in we get a correct statement $(0=0)$.

Note! $AB=0 \implies A=0$ or $B=0$!

Ex $x^2-5=0$ (try to factor).

$(x-\sqrt{5})(x+\sqrt{5})=0$

$x=\pm\sqrt{5}$

$x^2=k \implies x=\pm\sqrt{k}$

Method 2: Completing the square:

If we have an equation: $x^2+ax=b$ then turn this into $(x+c)^2=d$.

ie. $x^2+ax=b \Rightarrow ax=2xc \Rightarrow c=\frac{a}{2}$

Ex: $2x^2 - 3x - 4 = 0.$ | divide by 2

$x^2 - \frac{3}{2}x - 2 = 0.$ | add 2 to both sides

$$\underbrace{x^2 - \frac{3}{2}x} = 2.$$

$\hookrightarrow \underline{x^2 + 2ax} + a^2$

$2ax = -\frac{3}{2}x \Rightarrow a = -\frac{3}{4}$

$$\underbrace{x^2 - 2.\left(-\frac{3}{4}\right)x + \underline{\left(-\frac{3}{4}\right)^2}} = 2 + \underline{\left(-\frac{3}{4}\right)^2}$$

$$\left(x - \frac{3}{2}\right)^2 = 2 + \frac{9}{16} = \frac{41}{16} = \left(\frac{\sqrt{41}}{4}\right)^2$$ (take sqrt on both sides).

$$x - \frac{3}{2} = \pm \frac{\sqrt{41}}{4}$$

$$x = \pm \frac{\sqrt{41}}{4} + \frac{3}{2}.$$

Quadratic formula

There is a (faster) way to solve a quadratic eq.

$ax^2 + bx + c = 0$ has solutions

$\boxed{x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$. Furthermore $\boxed{ax^2 + bx + c = a(x - x_1)(x - x_2)}$.

Q: How did we obtain this formula?

A: By completing the square:

$$ax^2 + bx + c = 0 \Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

$$\Rightarrow x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + c - \left(\frac{b}{2a}\right)^2 = 0.$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - c = \frac{b^2 - 4ac}{4a^2}.$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}. \quad \blacksquare$$

Q: What happens when:

$b^2 - 4ac$ is :

- > 0 2 diff. solutions $\mathbb{R}$ solution
- < 0 $\sqrt{b^2 - 4ac}$ is an imag. # - $\mathbb{C}$ solutions
- $= 0$ $\Rightarrow x = \frac{b}{2a}$ - one solution

$$\Rightarrow a^2x + bx + c = a\left(x - \frac{b}{2a}\right)\left(x - \frac{b}{2a}\right)$$

$$= a\left(x - \frac{b}{2a}\right)^2.$$

the solutions are complex conjugates of each other!

Ex: $x^2 - 5 = 0$. we know $(x-\sqrt{5})(x+\sqrt{5})=0 \Rightarrow x_{1,2} = \pm\sqrt{5}$

but let's use the formula

$$ax^2+bx+c=0$$

$a=1$, $b=0$, $c=-5$.

$$x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{0 \pm \sqrt{0-4.1.(-5)}}{2.1} = \frac{\pm\sqrt{20}}{2} = \pm\frac{2\sqrt{5}}{2} = \pm\sqrt{5}$$

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(Inequalities).

Def: An inequality is a statement that two alg. are not equal in a particular way ie. one is bigger than the other.

eg. $3x+5 \geq 4x \iff 4x \leq 3x+5$.

Note: The solutions of an inequality are usually infinite #.

eg. $x>3$, solutions are all #s bigger than 3.

Q: Can it have only 1 solution?

Notation: for the set of mubers bigger than a we use the notation $(a, \infty) \leftarrow$ interval.

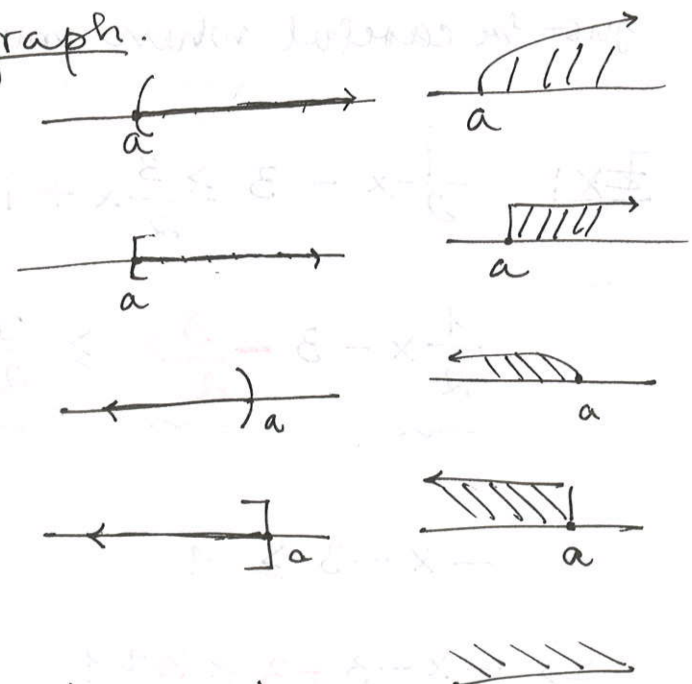
the Interval (a,b) contains all points smaller than b & greater than a.

if $b=\infty$ or $a=-\infty$ then we call the interval unbounded, otherwise we call the interval bonded.

to indicate greater or equal (resp. less or eq.) we use [bracket

For unbounded

Set	Interval notat	type	graph
$\{x \mid x>a\}$	(a,∞)	open	
$\{x \mid x\geq a\}$	$[a,\infty)$	closed from the left	
$\{x \mid x<a\}$	$(-\infty, a)$	open	
$\{x \mid x\leq a\}$	$(-\infty, a]$	closed from the right	
real #s $=\mathbb{R}$	$(-\infty, \infty)$.	open	



we never write $[-\infty, a]$, or $[a, \infty]$ but $(-\infty, a]$ or $[a, \infty)$.

Linear Inequalities (properties).

If $A < B$ then: (for some expression C).

- $A \pm C \leqslant B \pm C$
- $CA \leqslant BC$ (if $C > 0$)

  $CA \geqslant BC$ (if $C < 0$)
- $A/C \leqslant B/C$ (if $C > 0$)

  $A/C \geqslant B/C$ (if $C < 0$)

Ex: $A = 2 < 3 = B$, $C = -1$

$AC = -2$

$BC = -3$

$AC > BC$!

Solving linear inequalities: (the same way like equalities, just be careful when multiplying/div. by a negative #).

Ex: $\frac{1}{2}x - 3 \geqslant \frac{3}{2}x + 1$

$\frac{1}{2}x - 3 - \frac{3}{2}x \geqslant \underbrace{\frac{3}{2}x - \frac{3}{2}x}_{0} + 1$

$-x - 3 \geqslant 1$

$\Rightarrow -x \underbrace{-3 + 3}_{0} \geqslant 3 + 1$

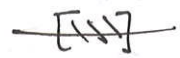
$-x \geqslant 4 \Rightarrow$ multip. by (-1) $\Rightarrow \boxed{x \leqslant -4}$

Compound inequalities:

Two (or more) linear inequalities connected with "and" or "or".

Ex: $5 \leq x \leq 10 \iff 5 \leq x$ and $x \leq 10$.

The solutions of those are often (not always) bounded intervals

Set	Interval	Graph.
$\{x \mid a < x < b\}$	(a, b)	
$\{x \mid a \leq x \leq b\}$.	$[a, b]$	
$\{x \mid a \leq x < b\}$	$[a, b)$	
$\{x \mid a < x \leq b\}$.	$(a, b]$	

Solving compound inequalities (with "and").

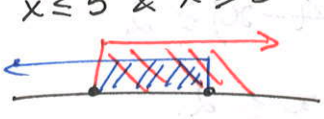
The way we solve these is we solve each linear ineq. in them then graph them and see where they have common solutions, that is where the graphs intersect.

Solving compound ineq. (with "or")

Again solve each individual linear ineq. an then graph the solutions. The solution to the compound is the set of solutions of the individual inequalities, that is the union of solutions: For union of two intervals we use "$\cup$" symbol.

Ex "and"

$x \leq 5$ & $x \geq 3$

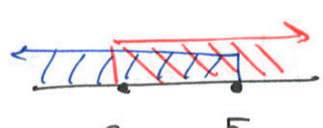


solution

$[3, 5]$

"or"

$x \geq 3$ or $x \leq 5$



solution

$(-\infty, \infty)$

"Every # is either less than 5 or bigger than 3"

Absolute value ineq. these we reduce to the compound ineq. with "or" or "and".

Ex: $|3x+2| \le 7$

Recall: If this was an equality:

$3x+2=7$ or $3x+2=-7$.

i.e. $3x+2>0$ i.e. $3x+2<0$.

$-7 \le 3x+2 \le 7$

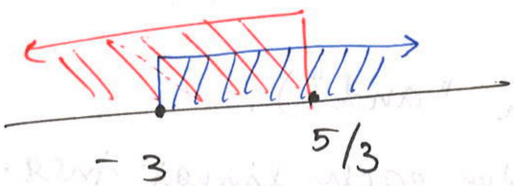
$-7 \le 3x+2$ and $3x+2 \le 7$

$3x \le 5$

$3x \ge -9$

$x \le 5/3$.

$x \ge -3$



solution $\left[-3, \frac{5}{3}\right]$

Ex: (clicker)?

$|3x| > -5$

$|3x|$ is alway $\ge 0 \Rightarrow$ for every x, $|3x| > -5$.

$\Rightarrow$ solution: every real # : $(-\infty, \infty)$.

Ex: $-2|4-x| \le -4$. $\cdot \mid -2$.

$\frac{-2}{-2}|4-x| \ge \frac{-4}{-2}$

$|4-x| \ge 2. \Rightarrow (4-x) \le -2$ or $(4-x) \ge 2$

$x \ge 6$ $x \le 2$.

$(-\infty, 2] \cup [6, \infty)$

Ex: $5|x-6|+3 \leq -2$

$5|x-6| \leq -5 \quad : 5\ (>0).$

$|x-6| \leq -1. \ < 0$! no solutions!

Ex: $-5|x-6|+3 \leq -2$

$-5|x-6| \leq -5 \quad : (-5) < 0.$

$|x-6| \geq 1.$

$(x-6) \geq 1$ or $x-6 \leq -1.$

$|\text{expr.}| \leq a$ vs $|\text{expr.}| \geq b$

$\Rightarrow -a \leq \text{expr.} \leq a$

$\Leftrightarrow -a \leq \text{expr.}$ and $\text{expr.} \leq a$

$\text{expr.} \geq b$ or. $\text{exp.} \leq -b$