Chapter 3

(3.1)

Note (Observation)

Let $f(x) = ax^2 + bx + c$ - a quadratic function.

Then we can always write $f(x)$, for some h & $k \in \mathbb{R}$,

$$f(x) = a(x-h)^2 + k.$$ (make a note of the signs!)

Q: why would we do this?

A: Now we see that $f(x)$ is obtained from the graph of x^2 by translation, reflection & scaling.

Ex: $f(x) = 2x^2 - 20x + 3$ $a=2$, $b=-20$, $c=3$

$= 2(x^2 - 10x) + 3$ complete the square for x^2-10x!

$= 2(x^2 - 10x + 25 - 25) + 3$

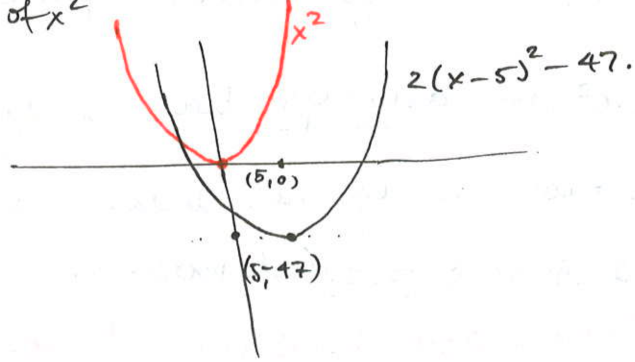
$= 2(x^2 - 10x + 25) - 50 + 3$

$= 2(x-5)^2 - 47$

translate down the graph of x^2 by 47 (vertical)

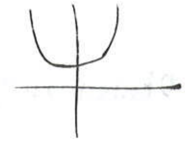
translate to the right by 5. (horizontal)

stretch graph of x^2

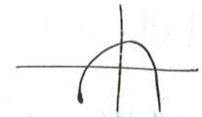


Some terminology about parabolas:

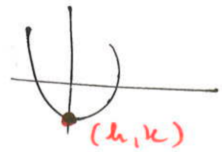
Let $f(x) = a(x-h)^2 + k$ be our function:



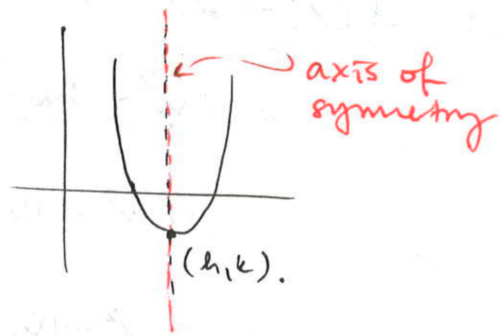
- if $a > 0$ the parabola <u>opens upwards</u>.



- if $a < 0$ the parabola <u>opens downwards</u>.



- (h, k) is the <u>vertex</u> of the parabola.
  $h = -b/2a \quad k = f(-b/2a)$.
- $f(x) = a(x-h)^2 + k$ is the <u>vertex form</u>
- $f(x) = ax^2 + bx + c$ is the <u>general form</u>.
- the vertical line on which (h,k) - the vertex - lies is called <u>axis of symmetry</u>



<u>Note</u>: If a parabola is open upwards the vertex is the <u>min. value</u> of the function

if it's open downwards $\Rightarrow$ the vertex is the <u>max. value</u>.

<u>Finding intercepts</u> for a quadratic function $f(x) = ax^2 + bx + c$

1. A quadratic function <u>always</u> has a y-intercept!
2. A quadratic function may not have an x-intercept.
   (there are 0, 1 or 2 x-intercepts.)

The x-intercepts have x-coordinates the solution of the equation $f(x) = 0$.

Quadratic inequalities (application).

Def: A quadratic inequality is an inequality of the form

$$ax^2+bx+c \gtreqless 0.$$

Method 1 (graphical) for solving a quadratic inequality $ax^2+bx+c \gtrless 0$.

1. turn it into equality, having 0 on one side & the polynomial expression on the other.

2. find the roots of this quadratic polynomial. ie. solve it, get x_1 & x_2 as solutions.

3. graph the ~~pa~~ function $f(x) = ax^2+bx+c$. (the x-intercepts are the points $(x_1, 0)$ & $(x_2, 0)$. where x_1 & x_2 are the roots).

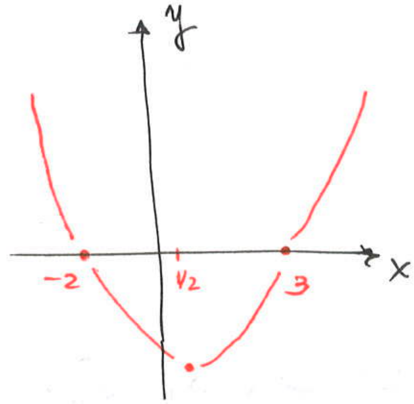
4. read the solutions from the graph of the parabola: $f(x)=y \Rightarrow$ ineq. is $y \lesseqgtr 0$.

Ex: $x^2 - x - 6 > 0$.

1. $x^2 - x - 6 = 0$

2. $x^2 - x - 6 = 0$ has roots $x_1 = 3$ & $x_2 = -2$.

3.



vertex (h,k) $\Rightarrow -\frac{b}{2a} = \frac{1}{2} = h$

4. when $x \in (-\infty, -2)$, $y>0$; $x \in (3, \infty)$, $y > 0$ $\Big\}$ $f(x) = y > 0$ for $x \in (-\infty, -2) \cup (3, \infty)$.
$x \in (-2, 3)$, $y < 0$

Method 2 (test points) for solving $f(x) = ax^2 + bx + c$.

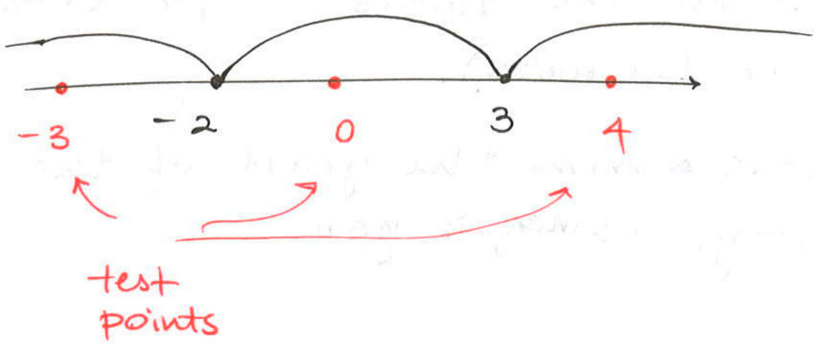
1. again get 0 on one side. turn into equality
2. find the roots of the equation.

   ↳ roots determine the intervals.

3. select a point in $\forall$ interval.
4. evaluate at the test point.
5. determine solution set.

Ex: $x^2 - x - 6 = 0$ $\qquad f(x) = y = x^2 - x - 6$.

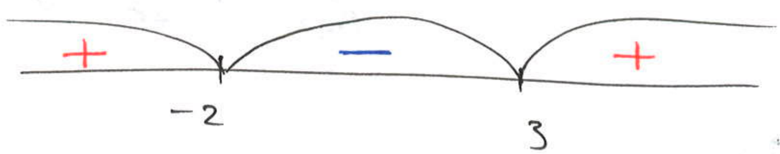
$x_1 = 3$, $x_2 = -2$.



$f(-3) = 9 + 3 - 6 = 11 \quad > 0$

$f(0) = -6 \quad < 0$

$f(4) = 16 - 4 - 6 = 6 > 0$.



$y = f(x) > 0$ for $x \in (-\infty, -2) \cup (3, \infty)$.

Note: these methods are equivalent!

Chapter 2.4

Def: Operations with functions:

$$(f \pm g)(x) = f(x) \pm g(x).$$

$$(f \cdot g)(x) = f(x)\,g(x)$$

$$(f/g)(x) = f(x)/g(x). \qquad (\text{for } g(x) \neq 0).$$

Ex: $f(x) = \sqrt{x} - 2$

$g(x) = x^2 + 5$

$$(g+f)(4) = f(4) + g(4) = (\sqrt{4} - 2) + (4^2 + 5) = 21.$$

$$(f/g)(1) = \frac{f(1)}{g(1)} = \frac{\sqrt{1} - 2}{1^2 + 5} = \frac{-1}{6}$$

Q: How are the domains of f & g related to the domains of $f \pm g$, fg, f/g.

A: For $f \pm g$, fg, f/g take the intersection of the domains of f & g & for f/g also exclude from the domain the points where $g(x) = 0$.

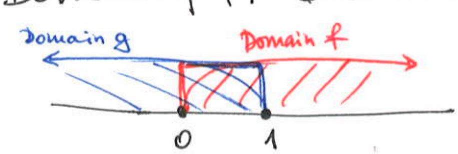
Ex: $f(x) = \sqrt{x}$, $g = \sqrt{1-x}$

Domain f: $[0, \infty)$

Domain g: $(-\infty, 1]$

$$\frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{1-x}}$$

Domain of f/g: $\{\text{Domain } f \cap \text{Domain } g\} - \{\text{zeroes of } g\} =$
 $x = 1$.



$= [0, 1)$

Composition of functions:

Def for two functions f & g, the composition of f & g is denoted $f \circ g$ & defined as

$$(f \circ g)(x) = f(g(x))$$

Note: we need $g(x)$ to be in the domain of f.
(b/c now $g(x)$ is the input for f!

Ex: evaluate: $f \circ g(4)$,
where $f(x) = \sqrt{x+2}$
$g(x) = x^2+7$.

$$(f \circ g)(4) = f(g(4)) = f(4^2+7) = f(16+7) = f(23)$$
$$= \sqrt{23+2} = \sqrt{25} = 5.$$

alternatively: compute first $f(g(x))$

$$f(g(x)) = \sqrt{g(x)+2} = \sqrt{(x^2+7)+2} = \sqrt{x^2+9}$$
$$\parallel$$
$$(f \circ g)(x)$$

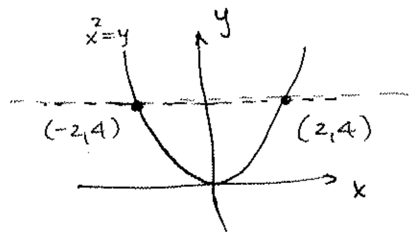
then $(f \circ g)(4) = \sqrt{4^2+9}$.

Chapter 2.5

Def: A function is one-to-one (also called injective)
If to every element of the domain it asigns a nique element of the range. No two elements of the domain are assigned the same element of the range.
i.e. no two ordered pairs with different x-coordinates have the same second coordinate.

Horitontal line test: A function is one-to-one if each horitontal line ~~that~~ y crosses its graph at no more than one point.

Ex.



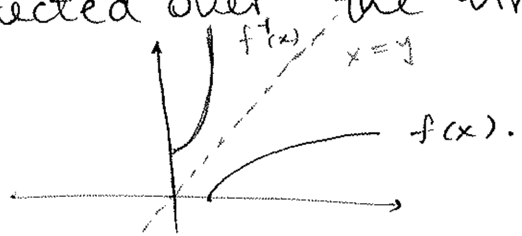
not 1 to 1.
we can see that if there is a hor. line crossing the graph the points where it crosses have the same y coordinates.

Def: The inverse of a one-to-one function f, is f^{-1} ("f inverse"). The domain of f^{-1} is the range of f.
The points on the graph of f^{-1} are obtained by exchanging x & y i.e. if (a, b) graph of $f \Rightarrow (b, a)$ is on the graph of f^{-1}.

Observation: Because of the way f^{-1} is defined, ~~then~~ we have that the graph of f^{-1} is the graph of f reflected over the line $x=y$.

Ex: $f(x) = \sqrt{x-1}$

$f^{-1}(x) = x^2+1$, $x \geq 0$



Finding $f^{-1}(x)$

1. Replace $f(x)$ by y
2. Interchange x & y.
3. Solve for y
4. Replace y by $f^{-1}(x)$.
5. Check that the domain of f = range of f^{-1} & that the domain of f^{-1} = range of f.

Ex: $f(x) = y = \dfrac{2x+1}{x-3}$ (1.)

$$x = \frac{2y+1}{y-3} \quad (2.)$$

$$x(y-3) = 2y+1. \quad (3)$$

$$xy - 3x = 2y + 1$$

$$xy - 2y = 3x+1.$$

$$y(x-2) = 3x+1$$

$$y = 3x+1/x-2.$$

$$f^{-1}(x) = \frac{3x+1}{x-2}. \quad (4)$$

(5) f^{-1}'s range = all $\mathbb{R}$ except 3 (b/c domain f is all $\mathbb{R} \setminus 3$).

Chapter 3

(3.2)

Zeroes of polynomial functions.

Def: Zero of a polynomial = root of polynomial = solut. of poly f-n.

Recall: for $P(x)$ - polynomial function, $Q(x)$ quotient

$$P(X) = Q(x).D(x) + R(x) \qquad \begin{array}{l} R(x) \text{ reminder} \\ D(x) \text{ divisor} \end{array}$$

$\deg D > \deg. R$.

Remainder thm

Let $D(x) = x - c$, then $\underline{P(c) = R(c)} = \underline{R}$.

b/c $P(c) = Q(c).D(c) + R(c)$, but $D(C) = c - c = 0$.

$\Rightarrow P(C) = R(c) = R$

as $\deg R < \deg D = 1 \Rightarrow \deg R = 0$. i.e. is a real number. ▨

Factor Thm

Let $P(x)$ have c as a zero (ie. a root or solution) if and only if $x - c$ is a factor of the polynomial.

b/c :. c is a zero of $P(x)$ if $P(c) = 0$, but by the previous thm. $P(C) = R \Rightarrow R = 0 \Rightarrow$

$P(x) = (x-c).Q(x) \Rightarrow (x-c)$ is a factor.

- if $P(x) = (x-c)\,Q(x) \Rightarrow P(c) = 0.Q(c) = 0$

$\Rightarrow C$ is a root. ▨

Fundam Thm. of Algebra

Every polynomial $P(x)$ can be factored completely over the complex numbers:

$$P(x) = (x-c_1)(x-c_2)\ldots(x-c_n), \quad \text{for } c_i \in \mathbb{C}.$$

Rational zero thm. Let $f(x) = a_n x^n + \dots + a_1 x + a_0$, where $a_i \in \mathbb{Z}$!,

$a_n \neq 0$ & $a_0 \neq 0$ and p/q is a zero of $f(x)$ ($p/q \in \mathbb{Q}$).

then p divides a_0 & q divides a_n.

(that is to say $a_0 = p \cdot k$ & $a_n = q \cdot m$ for some $k, m \in \mathbb{Z}$)

Recall: $\mathbb{Z}$ = integers
$\mathbb{Q}$ = rationals
"$\in$" is in ie. $a \in \mathbb{Z}$ means "a is an integer".

Find zeroes of a polynomial:

Ex: $f(x) = 2x^3 - 3x^2 - 11x + 6$

1. Find possible p/qs

try $\underbrace{\pm 1, \pm 2, \pm 3, \pm 6, \pm 1/2, \pm \frac{3}{2}}_{\text{these are possible } p/q\text{'s}}$.

$p = 1, 2, 3$ or 6
$q = 1$ or 2

2. plug them into $f(x)$

If $f(p/q) = 0$ then p/q is a zero.

Note: $x = 1/2, 3, -2$ all work.

3. divide by $(x - p/q)$

If in step 2. we find that $x = 1/2$ is a root

divide $f(x)$ by $x - 1/2$ to get

$$f(x) = (x - 1/2)(\underbrace{2x^2 - 2x - 12}_{f_1(x)}).$$

4. repeat for $f_1(x)$ (the quotient)

$f(x) = 2x^2 - 2x - 12 = 2(x^2 - x - 6)$.

factor $x^2 - x - 6$ ....

(recall that since this is a quadratic equation, we can just use the formula here).

Synthetic division (optional).

→ very useful for dividing by $x-c$.

Ex.

$$\underbrace{2x^3-5x^2+4x-6}_{P(x)} \;\Big|\; \underbrace{x-3}_{\text{Divisor } D(x)}$$

$$\underbrace{2x^2+x+7x}_{\text{quotient } Q(x)}$$

$$-\;(2x^3-6x^2)$$

$$x^2+4x$$

$$-\;(x^2-3x)$$

$$7x-6$$

$$-\;(7x-21)$$

$$\underbrace{15}_{R(x)=R}$$

usual division algorithm

For synthetic division:

step 1:

$$\underset{a_0 \text{ coeff. } D(x)}{3} \;\Big|\; \underbrace{2 \quad -5 \quad 4 \quad -6}_{\text{coeff's of } P(x)}$$

step 2: 2.1] bring down first coeff. = a_n

2.2] multiply it by a_0 from $D(x)$.

2.3] write answer beneath a_{n-1}.

2.4] add with a_{n-1}.

$$\begin{array}{c|cccc} 3 & 2 & -5 & 4 & -6 \\ & & 3.2 & & \\ \hline & 2 & 1 & & \end{array}$$

$1 = -5+3.2.$

step 3 continue (do step 2 again - only 2.2 - 2.4).

$$\begin{array}{c|cccc} 3 & 2 & -5 & 4 & -6 \\ & & 6 & 3.1 & 7.3 \\ \hline & 2 & 1 & 4+3 & 15 \end{array}$$

$2 \quad 1 \quad 4+3$: Quotient coefficients

15: Remainder

Ex: $f(x) = 3x^3 - 8x^2 - 8x + 8$

Find all the roots of the polynomial (real & imaginary)

Find possible Q. zeroes: try $\pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{8}{3}$

plug into $f(x)$. Recall if $f(c) = 0 \Rightarrow c$ is a root (zero).

here $f(\frac{2}{3}) = 0$.

divide by $(x - \frac{2}{3})$.

$$\begin{array}{c|cccc} \frac{2}{3} & 3 & -8 & -8 & 8 \\ & & 3 \cdot \frac{2}{3} = 2 & & \\ \hline & 3 & -6 & & \end{array}$$

Note for missing monomials in f write 0.
ie. if $f = x^3 + 1$
$\Rightarrow$ write $\begin{array}{|cccc} 1 & 0 & 0 & 1 \end{array}$
because $f = 1 \cdot x^3 + 0 \cdot x^2 + 0 \cdot x + 1$.

$$\begin{array}{c|cccc} \frac{2}{3} & 3 & -8 & -8 & 8 \\ & & 2 & \frac{2}{3}(-6) = -4 & \\ \hline & 3 & -6 & -12 & \end{array}$$

$$\begin{array}{c|cccc} \frac{2}{3} & 3 & -8 & -8 & 8 \\ & & 2 & -4 & \frac{2}{3} \cdot (-12) = -8 \\ \hline & 3 & -6 & -12 & 0 \end{array}$$

the remainder is 0. this means that f is divisible by $(x - 2/3)$ ie. $\frac{2}{3}$ is a root of $f(x)$, as expected.

$\Rightarrow f(x) = (x - \frac{2}{3})(x^2 - 6x - 12)$.

repeat for the quotient

ie. factor:

$x^2 - 6x - 12$ — this is quadratic eq. $\Rightarrow$

$x^2 - 6x - 12 = (x - x_1)(x - x_2)$, where $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$x_{1,2} = 1 \pm \sqrt{5}$.