Chapter 3
(3.3)

Def: Multiplicity of a root c of a polynomial $P(x)$ is the highest power to which $x-c$ appears in the factorization of $P(x)$.

Ex: $P(x) = (x-3)^{4}(x-2)^{2}(x-1)^{1}$

the roots are 3, 2 and 1
multiplicity of 3 is 4,
multiplicity of 2 is 2 &
multiplicity of 1 is 1.

Recall: (n-Root thm) we saw this as a corollary of the Fundamental thm. of algebra last time.
If $P(x)=0$ is a polynomial equation with (real or) complex coefficients, then, when multiplicity is considered $P(x)$ has n roots, where $n = \deg P(x)$.

Ex: $P(x) = (x-3)^{4}(x-2)^{2}(x-1)^{1}(x-5)^{1}$
then it has $4+2+1+1 = 8$ roots.

Thm (Conjugate Pairs) If $P(x)$ is a polynomial with real (not complex) coefficients, and $a+ib$ $(b \neq 0)$ is a root of $P(x)$ (ie. $P(a+ib)=0$) then $a-ib$ is also a root.

(b/c over the reals we have ax^2+bx+c as a factor of $P(x)$ & $D = b^2-4ac < 0$.)

Ex: $x^2 - 2x + 5 = 0$

$x = 1 \pm 2i$

Decartes's rule of Signs

Let $P(x)$ be a polynomial with real coefficients & with terms (monomials) written in descending order (of degree). Then:

- the number of positive roots of $P(x)$ is either equal to the sign changes (variations) of $P(x)$ or is less than that by an even number.
- the number of negative roots of $P(x)$ is equal to the sign changes (variations) of $P(-x)$ or is less than that by an even number.

Ex: $P(x) = 3x^4 - 5x^3 - x^2 - 8x + 4$

$+ \quad - \quad - \quad - \quad +$

$\Rightarrow$ two sign changes $\Rightarrow$ 2 or 0 positive roots

$$P(-x) = 3x^4 + 5x^3 - x^2 + 8x + 4$$

$+ \quad + \quad - \quad + \quad +$

$$= 3(-x)^4 - 5(-x)^3 - (-x)^2 - 8(-x) + 4$$

$\Rightarrow$ two sign changes $\Rightarrow$ 2 or 0 neg. roots.

! total 4 roots $\Rightarrow$ 2 post. & 2 neg. or
0 post. & 2 neg & 2 imaginary or
2 post. & 0 neg. & 2 imaginary or 4 imaginary.

Chapter 3

(3.4).

1. Factoring higher degree polynomials
(grouping, factoring out greatest common factor, substituting, using the rational zero theorem if applicable).

Ex: $x^3+3x^2+x+3=0.$ Factor this polynomial

Method 1: (Rational zero thm) applies since all coefficients are integers

Step1 possible rational roots are $\pm 3, \pm 1$

Step2 check if $f(3)=0$ or $f(-3)=0$. (can use Decartes's rule of signs)

$\Rightarrow$ we see that $f(-3)=0 \Rightarrow$ -3 is a root!

step3: divide x^3+3x^2+x+3 by $(x+3)=(x-(-3))$.

note the sign!

$$\begin{array}{r|rrrr} -3 & 1 & 3 & 1 & 3 \\ & & -3 & 0 & -3 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

correct!

$$\Rightarrow \quad x^3+3x^2+x+3=(x+3)\underbrace{(x^2+0x+1)}_{x^2+1.}$$

step4: factor the quotient:

$$x^2+1=0 \qquad x=\frac{-0\pm\sqrt{0^2-4.1.1}}{2}=\pm\frac{\sqrt{-4}}{2}=\pm i$$

$x=\pm i$

$\Rightarrow x^3+3x^2+x+3=(x+3)(x-i)(x+i).$

Method 2 (group).

$$\underbrace{x^3 + 3x^2}\underbrace{+ x + 3} = 0.$$

$$x^2(x+3) + (x+3) = 0.$$

$$(x+3)(x^2+1) = 0$$

$$(x+3)(x+i)(x-i) = 0.$$

Ex: $2x^5 = 16x^2$

$$2x^5 - 16x^2 = 0.$$

$$x^2(2x^3 - 16) = 0$$

$$2x^2(x^3 - 8) = 0$$

note that $x^3 - 8 = x^3 - 2^3$

$$2x^2(x-2)(x^2+2x+4) = 0.$$

2. Equations involving square roots: (always check your answer here)

(these are not polynomial equations!)

$\sqrt{2x+1} - \sqrt{x} = 1$

square each side;

$2x + \cancel{1} - 2\sqrt{2x+1}\cdot\sqrt{x} + x = \cancel{1}$

$3x = 2\sqrt{2x^2+x}$ square again!

$\left(\frac{3x}{2}\right)^2 = 2x^2 + x$

$\frac{9x^2}{4} = 2x^2 + x$

$9x^2 = 8x^2 + 4x \Rightarrow x^2 - 4x = 0$

$x = 0$ or 4

$\sqrt{x} = x - 2.$

square each side:

$x = (x-2)^2 = x^2 - 4x + 4$

$\Rightarrow x^2 - 5x + 4 = 0.$

$(x-4)(x-1) = 0$

$\Rightarrow x = 4$ or 1 but

$\sqrt{1} = 1 - 2 = -1$ incorrect! $\Rightarrow$ only 4 is root

Note: Always plug in the roots you obtained to see if they actually satisfy the original equation!

3. Equations with rational exponents:

$x^{-3/2} = 1/8$ (raise each side to the power $-2/3$)

$\left(x^{-3/2}\right)^{(-2/3)} = (1/8)^{-2/3}$

$x^{\left(-\frac{3}{2}\right)\cdot\left(-\frac{2}{3}\right)} = \dfrac{1}{8^{-2/3}}$

$x = 8^{2/3} = \sqrt[3]{8^2} = \left(\sqrt[3]{8}\right)^2 = 2^2 = 4.$ check your answer!

Note: $x^{-3/2} = \sqrt{\dfrac{1}{x^3}}$ note that $x \gneq 0$ for this to make sense. $4 \gneq 0$ ✓ $\Rightarrow$ 4 is a root.

Strategy: for $x^{m/n} = a$

- raise both sides to the power (n/m) to get

$x^{m/n \cdot n/m} = a^{n/m}$

$x = a^{n/m}$.

- Check your answer if n was even!

(note that if $a < 0$ & n is even - no solution)

4. Rational exponents but of quadratic type:

Ex: $x^{2/3} - 9x^{1/3} + 8 = 0$

$x^{1/3} = u$ substitute:

$u^2 - 9u + 8 = 0$. $\Rightarrow u = 8$ or 1.

$= (u-8)(u-1) = 0$.

.... we are not done ..... (substitute back)

$x^{1/3} = 8$ or $x^{1/3} = 1$

$x^{1/3 \cdot 3/1} = 8^{3/1}$ $\qquad$ $x^{1/3 \cdot 3/1} = 1^{3/1}$

$x = 8^3 = 512$ $\qquad$ $x = 1$

5. Equations involving abs. value:

Ex: $|x^2 - 6| = 5x$

5x has to be positive, because $|x^2-6| \geq 0$.

$\Rightarrow x \geq 0$

$\Rightarrow x^2 - 6 = 5x$ or $x^2 - 6 = -5x$ (if $x^2 - 6 < 0$)

$x^2 - 5x - 6 = 0$ $\qquad$ $x^2 + 5x - 6 = 0$

$(x-6)(x+1) = 0$ $\qquad$ $(x+6)(x-1) = 0$

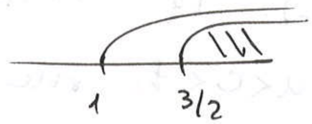
$x = 6$, $x = -1$ ✗ $\qquad$ $x = -6$ ✗ or $x = 1$ ✓

$x = 6$ or $x = 1$

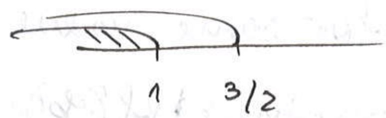
<u>Ex</u> $|a-1| = |2a-3|$ Solve.

Note that if two expressions have the same abs. value, then they either ~~have~~ have the <u>same</u> or <u>opposite</u> signs:

1) $\begin{cases} a-1>0 \\ 2a-3>0 \end{cases}$ $\begin{cases} a>1 \\ a>3/2 \end{cases}$



or $\begin{cases} a-1<0 \\ 2a-3<0 \end{cases}$ $\begin{cases} a<1 \\ a<3/2 \end{cases}$



same signs

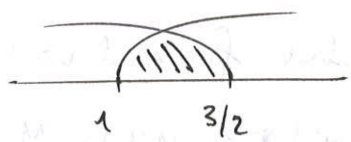
we get

$(a-1) = (2a-3)$

$a-1 = 2a-3$

$\underline{a=2}$. – this is the case when $a > 3/2$ ✓

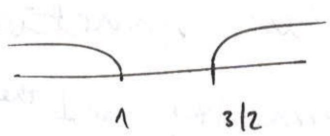
2) $\begin{cases} a-1>0 \\ 2a-3<0 \end{cases}$ $\begin{cases} a>1 \\ a<3/2 \end{cases}$



different signs.

or

$\begin{cases} a-1<0 \\ 2a-3>0 \end{cases}$ $\begin{cases} a<1 \\ a>3/2 \end{cases}$



no solut.

we get

$(a-1) = -(2a-3)$

$a-1 = -2a+3$

$\underline{a = \frac{4}{3}}$

$4/3 \in \left(1, \frac{3}{2}\right) \Rightarrow$ is a solution ✓

Chapter 3

(3.5).

Theorem: The Intermediate value thm. (IVT) - without proof (for polynomial functions).

Let f be a polynomial function & $[a, b]$ an interval s.t. $f(a) \neq f(b)$. If k is such that $f(a) < k < f(b)$, then exists a c, $a < c < b$ such that $f(c) = k$.

Application: if for some value a, $f(a)$ is positive & for another value b, $f(b)$ is negative, then there is a value c, $a < c < b$ st. $f(c) = 0$.

Sketching the graph of a polynomial function:

1. Symmetry:
   - check if the function is even
     → symmetry wrt y-axis
   - check if the function is odd
     → symmetry wrt the origin
   - check if quadratic
     → symmetry wrt its axis of symmetry. (duh!)

2. Behaviour at x-intercepts (find these)

   (Recall these are the 0-s of $f(x)$ if $f(x)$ is a polynomial).

<u>Leading coeff. test</u> Let $y = f(x) = a_n x^n + \dots + a_1 x + a_0$ then

for n-odd, $a_n > 0$: $(x \to \infty, y \to \infty)$ & $(x \to -\infty, y \to -\infty)$

eg. $y = f(x) = 2x^3$ ie. $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to -\infty} f(x) = -\infty$

for n-odd, $a_n < 0$: $(x \to \infty, y \to -\infty)$ & $(x \to -\infty, y \to \infty)$

eg. $y = f(x) = -2x^3$ ie. $\lim_{x \to \infty} f(x) = -\infty$ $\lim_{x \to -\infty} f(x) = \infty$

for n-even, $a_n > 0$: $(x \to \infty, y \to \infty)$ & $(x \to -\infty, y \to +\infty)$

eg. $y = f(x) = 3x^4$ $\lim_{x \to \infty} f(x) = \infty$ $\lim_{x \to -\infty} f(x) = \infty$

for n-odd, $a_n < 0$: $(x \to \infty, y \to -\infty)$ & $(x \to -\infty, y \to -\infty)$

eg: $y = f(x) = -3x^4$ $\lim_{x \to \infty} f(x) = -\infty$ $\lim_{x \to -\infty} f(x) = -\infty$

<u>Hint:</u> think of $f(x) =$ leading monomial $a_n x^n$ & see what happens then.

* Graphing polynomial functions $f(x)$ (strategy)

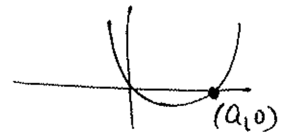
1. check for symmetry – odd, even, quadratic.
2. find zeroes (real) of $f(x)$
3. behaviour at x-intercepts (graph of $f(x)$ touches, crosses the x-axis).
4. determine behaviour at $x \to \pm\infty$
5. calculate y-intercepts & a few more pairs
6. connect continuously the pts.

~~Let~~ Observe

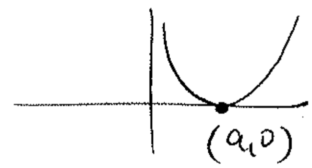
a be a root of $f(x)$ ← polynomial function.

& let k be the multiplicity of a.

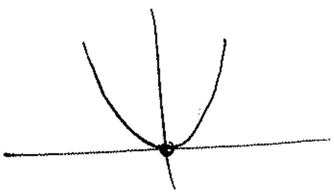
(ie $f = (x-a)^k(\ldots\ldots)$.)

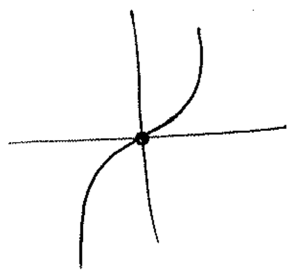


- if k is odd ⇒ graph crosses the x-axis at $(a,0)$
- if k is even ⇒ graph touches the x-axis at $(a,0)$.



Ex: $f = x^2$ & $f = x^3$





$k=3$
$a=0$

$k=2$
$a=0$

3. Leading coeff. test (what happens when $x \to \pm\infty$).

(no precise def. of limit - only intuition).

Ex: $y = x^3 - x + 1$

for $x \to \infty$ ie. $\lim_{x\to\infty} x^3 - x + 1$ depends really on the leading term $= x^3$

"$f(x)$ behaves like x^3 when $x \to \pm\infty$" (ie. the one with highest power of x).

⇒ here as x becomes very big positive y is $\to \infty$

as x — " — negative $y \to -\infty$

Ex: Sketch: $f(x) = x^3 - 5x^2 + 7x - 3$

1. – not quadratic: (has x^3 & x^2 ....)
   – not even $\Big\}$ $f(-x) = -x^3 - 5x^2 - 7x - 3$
   – not odd $\quad\neq f(x)$
   $\quad\neq -f(x)$.

2. By Decartes's rule of signs:

$f(x) = x^3 - 5x^2 + 7x - 3$ 3 or 1 positive

+ – + –

$f(-x) = -x^3 - 5x^2 - 7x - 3$ 0 neg.

– – – –

By rational zero theorem:

$p/q = \pm 3$ or ± 1 but by the previous $\Rightarrow$ no neg. roots

$\Rightarrow$ check $+3$ & $+1$.

$f(1) = 0$

Divide out by $x - 1$

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 7 & -3 \\ & & 1 & -4 & 3 \\ \hline & 1 & -4 & 3 & 0 \end{array}$$

$\Rightarrow f(x) = (x-1)\underbrace{(x^2 - 4x + 3)}_{(x-3)(x-1)} = (x-1)^2(x-3)$.

3. $f(x) = (x-1)^2(x-3)$.

root 1, multip. 2 $\Rightarrow$ graph touches x-axis at 1

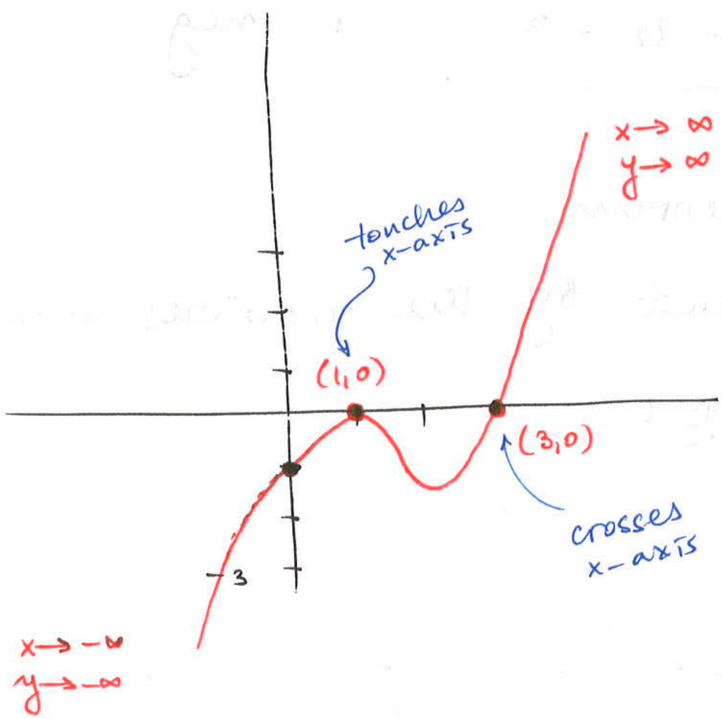
root 3, multip. 1 $\Rightarrow$ graph crosses x-axis at 3

4. when $x \to \infty$ f(x) behaves like x^3 $y \to \infty$

$x \to -\infty$ — " — $y \to -\infty$

i.e. $\lim_{x\to\infty} f(x) = \infty$ & $\lim_{x\to -\infty} f(x) = -\infty$.

5. y-intercept : $x = 0 \Rightarrow y = -3$.



Applicat: Polynomial inequalities:

Let $f(x) = x^4 + x^3 - 15x^2 - 3x - 36 \underline{<} 0.$ Solve.

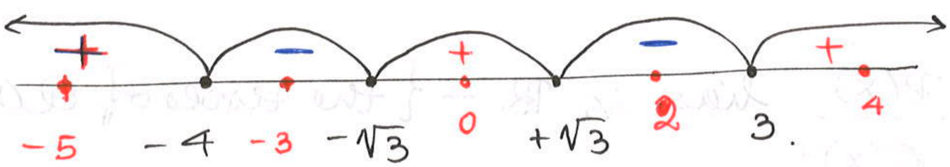
(find for which x, $f(x) < 0$)

solve:

$f(x) = 0. \Rightarrow$

$f(x) = (x-3)(x+4)(x^2-3).$

$= (x-\underline{3})(x+\underline{4})(x+\underline{\sqrt{3}})(x-\underline{\sqrt{3}}).$



$f(-5) = 176$ +

$f(-3) = -36$ −

$f(0) = 36$ +

$f(2) = -6$ −

$f(4) = 104$ +

$f(x) < 0$ when we are in a "−" interval:

$x \in (-4, -\sqrt{3}) \cup (\sqrt{3}, 3)$

for $f(x) \leq 0$

$x \in [-4, -\sqrt{3}] \cup [\sqrt{3}, 3]$.

Chapter 3
(3.6)

Def: Rational function is the quotient of two polynomial functions $P(x)$ & $Q(x)$, s.t. $Q(x) \neq 0$.

$$f(x) = \frac{P(x)}{Q(x)}.$$

Note The domain of $\frac{f(x)}{g(x)}$ is $\{$the domain $f(x)\} \cap \{$domain $g\}$ $- \{$the zeroes of $g\}$

but $P(x)$ as a polynomial has domain $\mathbb{R}$ & $Q(x)$ too $\Rightarrow$

domain of $\frac{P(x)}{Q(x)} = f(x)$ is $\mathbb{R} - \{$ the zeroes of $Q(x)\}$

Ex: $g(x) = \frac{2x-3}{x^2-4}$ $\Rightarrow$ domain with $g(x)$ is the

set of x for which $x^2 - 4$ isn't 0.

$(x^2-4) = (x-2)(x+2) \Rightarrow$ $x = \pm 2$.

$\Rightarrow$ domain is $\mathbb{R}$ exc. ± 2.

Def: Let $f(x) = \frac{P(x)}{Q(x)}$ be a rational function.

- Then: if $|f(x)| \to \infty$ as $x \to a$ then $x = a$ is a vertical asymptote. i.e. $\boxed{\lim_{x \to a} |f(x)| = \infty}$
- if $f(x) \to a$ as $x \to \infty$ (or $x \to -\infty$) the line $y = a$ is a horizontal asymptote. i.e. $\boxed{\lim_{x \to \pm\infty} f(x) = a}$
- there are also slant (or oblique) asymptotes. (see example).

Ex Find the asymptotes of:

$$f(x) = \frac{2x^2+3x-5}{x+2} = \frac{P(x)}{D(x)}.$$

- as $x \to -2$ we have $x+2 \to 0$ then (ie) $D(x) \to 0$

  but then $f(x) \to \infty$ $\left(= \frac{\text{something}}{0}\right)$.

  $\Rightarrow x = -2$ is a vertical asymptote.

- note that $\deg D(x) < \deg P(x) \Rightarrow$ we can divide.

$$\begin{array}{r|rrr} -2 & 2 & 3 & -5 \\ & & -4 & 2 \\ \hline & 2 & -1 & -3 \end{array}$$

$$f(x) = 2x - 1 + \frac{(-3)}{x+2}.$$

Important to remember

$\lim_{b \to \infty} \frac{a}{b} = 0$ $\quad a \neq 0, \infty$

$\lim_{b \to 0} \frac{a}{b} = \infty$ $\quad a \neq 0, \infty$

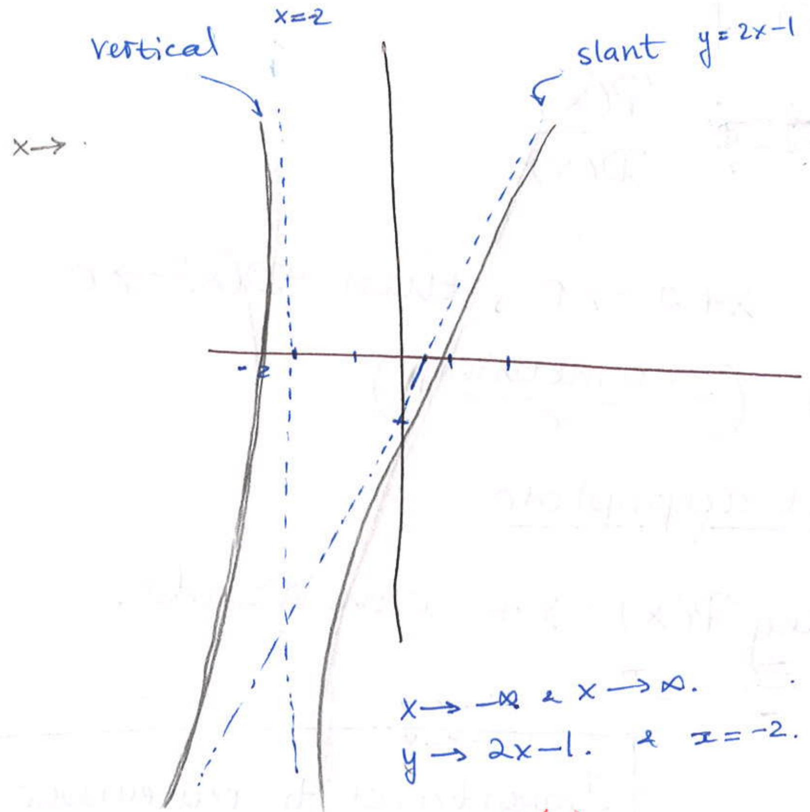
if $|x| \to \infty$, $\frac{-3}{x+2} \to 0$

thus as $|x| \to \infty$, $f(x) \to 2x - 1$.

the line $2x-1$ is a slant ~~assymptote~~ asymptote.

Graph the funct. $f(x) = 2x^2 + 3x - 5 / (x+2)$

(the procedure is the same as for polyn. functions but take into account the asymptotes this time).



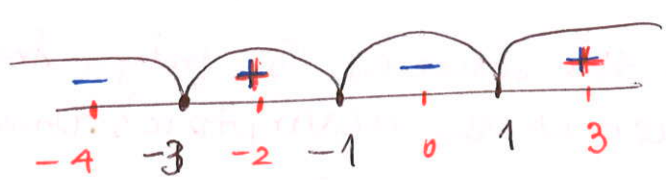
- The graph of $f(x) = \dfrac{P(x)}{Q(x)}$ never crosses ~~the~~ an asymptote!

- check symmetry none other than wrt asymptotes
- check for ($x \to \infty$, $x \to -\infty$) & $x \to$ asymptotes!
- find intercepts x-intercepts: when $P(x)=0$ are $(1,0)$ $(-2{,}5, 0)$.
  y-intercept $(0, -2{,}5)$.
- plot & connect

Rational inequalities

Solve: $\dfrac{x+3}{x^2-1} \geq 0$ this is $\dfrac{P(x)}{Q(x)} = \dfrac{x+3}{(x-1)(x+1)} \geq 0$.

- Find intervals = determined by 0s of $P(x)$ & $Q(x)$



can't include these, they are not in the domain.

solution: the "+" intervals $[-3, -1) \cup (1, \infty)$

Example

Sketch the graph of the polynomial: $f(x) = x^3 - 5x + 2$.

1. check symmetry:

$$f(-x) = -x^3 + 5x + 2 \qquad f(x) \neq f(-x)$$

$$-f(x) = -x^3 + 5x - 2. \qquad f(-x) \neq -f(x)$$

$\Big\}$ no symm.

2. Find x-intercepts: (= possible roots of $f(x)$).

possible rational roots:

$$\pm 2, \ \pm 1.$$

check: $f(1),\ f(-1),\ f(-2) \neq 0 \qquad f(2) = 0$!

$$\Rightarrow f(x) = (x-2)\,Q(x).$$

to find $Q(x)$ divide $f(x)$ by $(x-2)$:

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -5 & 2 \\ & & 2 & 4 & -2 \\ \hline & \underbrace{1 \quad 2 \quad -1}_{Q(x)} & & & 0 \end{array}$$

$$Q(x) = 1 \cdot x^2 + 2 \cdot x + (-1) = x^2 + 2x - 1.$$

factor $Q(x)$: $\quad x^2 - 2x - 1 = (x - x_1)(x - x_2)$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = 1 \pm \sqrt{2}.$$

$\Rightarrow f(x) = (x-2)(x-(1+\sqrt{2}))(x-(1-\sqrt{2}))$.

the x-intercepts are:

$(2, 0)$

$(1 \pm \sqrt{2}, 0)$

- they are all with multip. 1. $\Rightarrow$ graph crosses at these points.

- $x \to \infty \quad y \to \infty$

  $x \to -\infty \quad y \to -\infty$

  (behaves like x^3). when $x \to \pm\infty$

- y-intercepts: $x=0 \quad y = f(0) = 2$.