Exponential & Logarithmic functions
(Chapter 4)
4.1

Definition Exponential function with base a is $f(x) = a^x$, where $a > 0$ & $a \neq 1$. (x is any real number)

Ex: $f(x) = 2^x$, evaluate: $f(\frac{1}{2})$, $f(2)$, $f(-1)$

$$f(\tfrac{1}{2}) = 2^{1/2} = \sqrt{2}$$

$$f(2) = 2^2 = 4.$$

$$f(-1) = 2^{-1} = \frac{1}{2}$$

Domain of a^x (for $a > 0$, $a \neq 1$) is $\mathbb{R}$.

Question: why are we excluding 1?

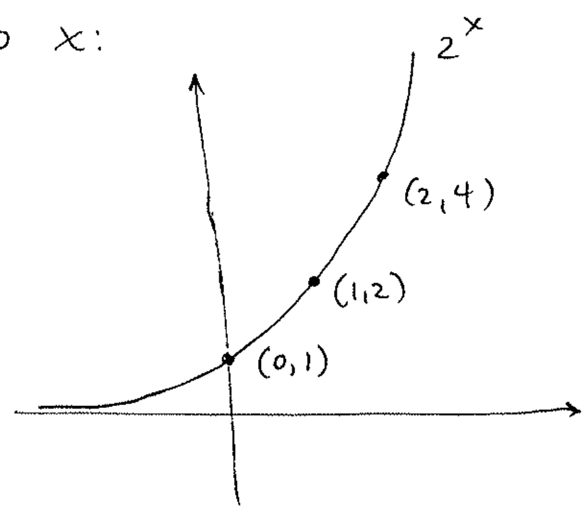
Answer: $1^x = 1$ for every x – it's the constant function.

Graph of a^x, when $a > 1$

Ex: $2^x = f(x)$

To graph give a few values to x:

x	-2	-1	0	1	2
$f(x) = y$	$1/4$	$1/2$	1	2	4

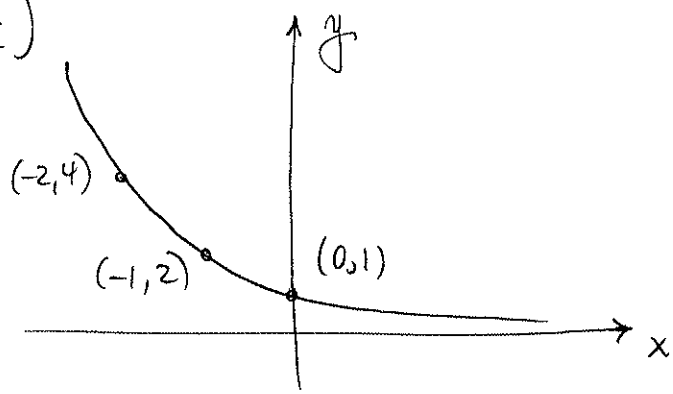


Observe: the bigger the a, the steeper the funct. graph.
(the function grows faster for bigger a's).

Graph of expon. a^x $(0<a<1)$.

Ex. let $a=\frac{1}{2}$ $\Rightarrow$ $f(x)=\left(\frac{1}{2}\right)^x$

x	-2	-1	0	1	2
$f(x)=y$	4	2	1	$1/2$	$1/4$



Observe:

1) $f=(\frac{1}{2})^x$ has graph that is the reflection of the graph of $f=(2)^x$ with respect to the y-axis!

2) if $a>0$ the function $f=a^x$ is increasing
if $0<a<1$ the function $f(x)=a^x$ is decreasing.

3). the ~~a~~ x-axis is an asymptote (horizontal).

4) the y-intercept is always $(0,1)$, because $a^0=1$.

5) the domain is $\mathbb{R}$, but the range is $(0,\infty)$.

6) the function $f(x)=a^x$ is one to one. (do the horizontal line test).

Exponential equations

Ex: $2^x = 2^3 \Rightarrow \underline{x=3}$.

Note: We can solve the above equation like this, because we know that $f(x)=a^x$ is one to one, i.e.

$$f(x_1) = f(x_2) \text{ if and only if } x_1 = x_2.$$

$$\text{i.e. } a^{x_1} = a^{x_2} \Leftrightarrow x_1 = x_2.$$

Ex: $\left(\frac{1}{10}\right)^x = 100 \Rightarrow 10^{-x} = 10^2 \Rightarrow \underline{x=-2}$

Def: The irrational number e ($\approx 2.7182\ldots$) is defined as

$$e = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

It is very often used as base for the exponential function.

Chapter 4.2

Def: For $a>0$, $a\neq 1$ the logarithmic function with base a is denoted by $f(x) = \log_a(x)$, where

$$y = \log_a x \text{ if and only if } a^y = x.$$

Notation: common logarithmic function $y = \log x \Leftrightarrow 10^y = x$

natural logarithmic function $y = \ln x \Leftrightarrow e^y = x$

Ex.: Compute: $\ln(1) = ?$, $\log_3(9) = ?$

- $e^0 = 1 \Rightarrow \ln(1) = 0.$

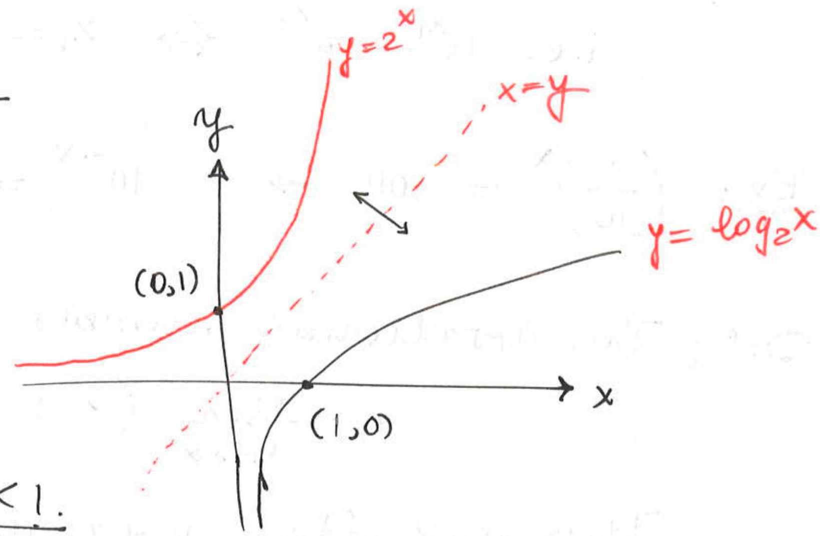
- $\log_3 9 = 2$ because $3^2 = 9$.

Observe: The log function is the inverse of the exponential! We can see this from the graphs:

Graph of $\log_a(x)$, for $a > 1$

Ex: $f(x) = \log_2(x)$

x	$1/2$	1	2	4
$y = f(x)$	-1	0	1	2

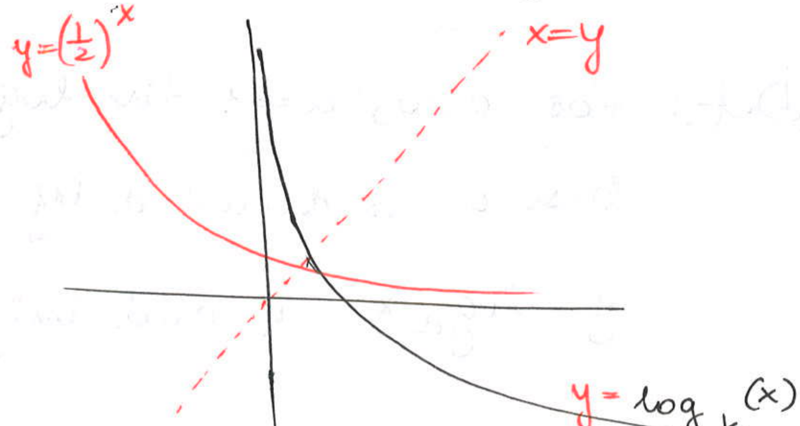


Graph of $\log_a(x)$, for $0 < a < 1$.

Similarly for $f(x) = \log_{\frac{1}{2}}(x)$

We can see that the graph of $\log_{\frac{1}{2}}(x)$ is symmetric to the graph of $\left(\frac{1}{2}\right)^x$ with respect to the $x=y$ line.

x	4	2	1	$1/2$	$1/4$
$f(x) = y$	-2	-1	0	1	2



Observe: 1) $f(x) = \log_a(x)$ is increasing for $a > 1$ and decreasing for $0 < a < 1$.

2) the x-intercept is always $(1,0)$.

3) the graph of $f(x) = \log_a x$ has a vertical asymptote - the y-axis.

4) domain is $(0,\infty)$ (= range of a^x) and range is $(-\infty,\infty)$ (= domain of a^x).

5) $\log_a x$ is one-to-one.

Logarithmic equations: (we can use observat. #5)

for $a > 0$, $a \neq 1$ $\log_a(x_1) = \log_a(x_2)$ iff $x_1 = x_2$.

Ex: $\log_3(x) = -2$

$\Rightarrow x = 3^{-2} = 1/9.$

Ex: $\log_x 5 = 2$

$x^2 = 5$

$x = \pm\sqrt{5}.$

Chapter 4.3

<u>Recall</u>: that $\log_a x$ is the inverse of a^x.

• Then: for $a>0$ & $a \neq 1$

$$\boxed{\begin{aligned} \log_a(a^x) &= x \\ a^{\log_a x} &= x \end{aligned}}$$

b/c $f^{-1}(f(x)) = x$.

b/c $f(f^{-1}(x)) = x$.

<u>Ex</u>: $e^{\ln x^2} = x^2$

$\log_7(7^{(2x-1)}) = 2x-1$.

• Product rule for logarithms:

for $M, N > 0$

$$\boxed{\log_a(MN) = \log_a M + \log_a N}$$

<u>Proof</u>: $a^{\log_a M + \log_a N} = a^{\log_a M} \cdot a^{\log_a N} = M.N = a^{\log_a (MN)}$

but $a^{x_1} = a^{x_2} \Rightarrow x_1 = x_2 \Rightarrow \log_a M + \log_a N = \log_a MN$. ∎

<u>Ex</u>: $\log_3(x) + \log_3(6) = \log_3(6x)$

<u>Note</u>: the base of the log's have to be the same, otherwise you can't add.

- Quotient rule for logarithms:

for $M>0$, $N>0$

$$\boxed{\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N}$$

Pf: Like in the product case:

$$a^{\log_a\left(\frac{M}{N}\right)} = M/N$$

$$a^{\log_a M - \log_a N} = \frac{a^{\log_a M}}{a^{\log_a N}} = \frac{M}{N}.$$

& $\log_a x$ is one to one $\Rightarrow \log_a M/N = \log_a M - \log_a N$. ∎

- Logarithm power rule:

for $M>0$ & any $N \in \mathbb{R}$ (i.e. N real number).

$$\boxed{\log_a M^N = N(\log_a M)}$$

Ex: $\log 3^8 = 8 \log 3$ (recal $\log 3 = \log_{10} 3$).

Note also: $\boxed{\begin{array}{l}\log_a a = 1 \\ \log_a 1 = 0.\end{array}}$ & in partic: $\boxed{\begin{array}{l}\ln(e) = 1 \\ \ln(1) = 0\end{array}}$.

Ex: Rewrite as a single logarithm:

$$\underbrace{\ln(x-1) + \ln 3}_{} - \underbrace{3 \ln x}_{} = \ln(3x-3) - \ln x^3$$
$$= \ln\left(\frac{3x-3}{x^3}\right).$$

- Base-change formula

for $a > 0$ $\quad a \neq 1$ $\quad M > 0$
$\quad b > 0$ $\quad b \neq 1$

$$\boxed{\log_a M = \frac{\log_b M}{\log_b a}}$$

Pf: Let $a^x = M \iff x = \log_a M$.

then $\log_b(a^x) = \log_b(M)$ (take $\log_b$- on both sides)

$$x \log_b a = \log_b M$$

$$\Rightarrow x = \frac{\log_b M}{\log_b a}$$

$$\Rightarrow \frac{\log_b M}{\log_b a} = \log_a M \quad ◼$$