Chapter 4.
(4.4)

Solving more logarithmic and exponential equations:

1. Equation involving a single logarithm: -(use def. of log)

Ex: $\log(x-3)=4$, $x=?$ recall: $\log_a x = y \Leftrightarrow a^y = x$.

& $\log x = \log_{10}(x)$.

$\Rightarrow x-3 = 10^4$

as here $a=10$

$x = x-3$

$y=4$

$\Rightarrow x = 10^4+3 = 10{,}003$.

2. Equations with more than one logarithm (with the same base)

(use the log. product, quotient & power rule to write the equation as an eq. of the prev. type i.e. with a single log.).

Ex: $\underbrace{\log_2 x + \log_2 (x+2)}_{\text{product rule}} = \log_2 (6x+1)$. find x.

$\log_2(x(x+2)) = \log_2(6x+1)$. use the one-to-one property for logarithms.

(recall: $\log_a x_1 = \log_a x_2$ if and only if $x_1 = x_2$)

in our case $a=2$, $x_1 = x(x+2)$ & $x_2 = 6x+1$).

$\Rightarrow x(x+2) = 6x+1$.

$x^2+2x-6x-1=0$

$x^2-4x-1=0$. $\Rightarrow x = 2\pm\sqrt{5}$.

3. Exponential equations involving a single exponential expression
(take logarithm on both sides)

Ex: $3^{2t-1} = 5$, find t.

use the def. $a^x = y \Leftrightarrow \log_a y = x$
(ie. "take log on both siges of the egality"
ie. for the equation $a^x = y$ we get:
$\log_a a^x = \log_a y \Rightarrow x = \log_a y$.).

for this example $a = 3$
$x = 2t-1$
$y = 5$

$\Rightarrow x = \log_a y$ ie. $2t - 1 = \log_3 5$.

$\Rightarrow 2t = \log_3 5 + 1$

$t = \dfrac{\log_3 5 + 1}{2}$.

4. Equations involving two exponential expressions.
(take log, ln or $\log_a$ on both sides)

Ex: $6^x = 7^{x-1}$, find x.

- take ln on both sides

$\Rightarrow \ln 6^x = \ln 7^{x-1}$. (recall this is the same as $\log_e 6^x = \log_e 7^{x-1}$).

- use the power rule for log.

$\Rightarrow x \ln 6 = (x-1) \ln 7$. $\Rightarrow x \ln 6 = x \ln 7 - \ln 7$

$$x \ln 6 = x \ln 7 - \ln 7.$$

$$x \ln 6 - x \ln 7 = -\ln 7$$

$$x(\ln 6 - \ln 7) = -\ln 7$$

$$x = -\frac{\ln 7}{\ln 6 - \ln 7} = \frac{-\ln 7}{\ln 6/7}.$$

note that it doesn't matter if in the beginning of the problem you take ln or log or $\log_2$ - ...
the result will be the same (remember the change of base formula).

<u>Ex</u>: $6^{5x-1} = 2^x$, find x.

take $\log_2$ - here (note it can be ln, log, or $\log_3$ or any other log instead).

$$\Rightarrow \log_2 6^{5x-1} = \log_2 2^x$$

$$\Rightarrow (5x-1)\log_2 6 = x$$

$$5x\log_2 6 - \log_2 6 = x$$

$$5x\log_2 6 - x = \log_2 6.$$

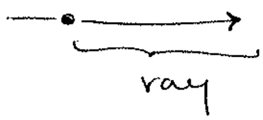
$$x(5\log_2 6 - 1) = \log_2 6$$

$$x = \log_2 6 \Big/ 5\log_2 6 - 1.$$

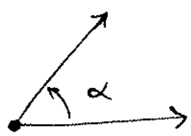
Chapter 5
Trigonometric functions.

Some definitions:

- a ray – a point on a line together with the points on the line on the side of that pt.

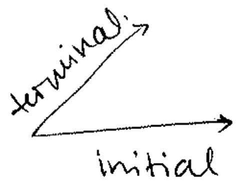


- an angle – the union of two rays (~~at~~ with) a common vertex.

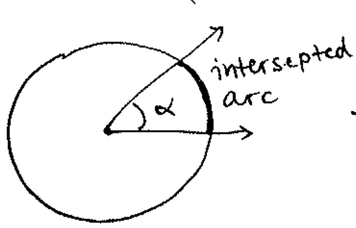


angle α

- we sometimes think of an angle as obtained from rotating one ray away from a fixed ray in a direct. α. we call the initial ray – initial side of the angle & the rotated one – terminal side



- if the vertex of an angle is the center of a circle:

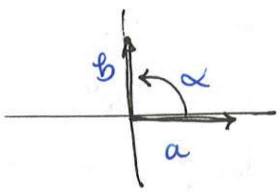


then the piece of the circle that's between the two rays of the angle is called intercepted arc.

- an angle is in standard position if it's vertex is the origin of the x-y plane.

<u>Def</u>: The <u>measure</u> of an angle α indicates the amount of rotation of the terminal side from the initial position. (this is found using any circle centered at the vertex). If we rotate to complete a whole circle i.e. we arrive at the initial ray then we say we revolved $360°$. (degrees).
The degree measure of an angle = degrees in the intercepted arc of the circle.
<u>note</u>: the degree measure is positive when going (rotating) counterclockwise & negative when going clockwise.

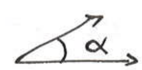
<u>Ex</u>:



from ray a to ray b we travelled conterclockwise for 1/4 of a full revolution
$\Rightarrow$ the angle α has measure $\frac{1}{4} \cdot 360°$
$= 90°$.

<u>More definitions</u>:

- <u>acute</u> angle - has measure between 0 & $90°$

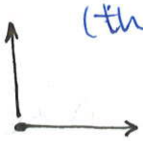


- <u>right</u> angle - has measure $90°$

- <u>obtuse</u> angle - has measure between $90°$ & $180°$.

- An angle in standard position is said to lie in a quadrant, ~~if~~ where it's terminal side lies. If the terminal side is on an axis, then it's called a <u>quadrantal</u> angle.
(they don't lie in any quadrant)

<u>Ex</u>:



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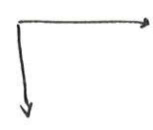
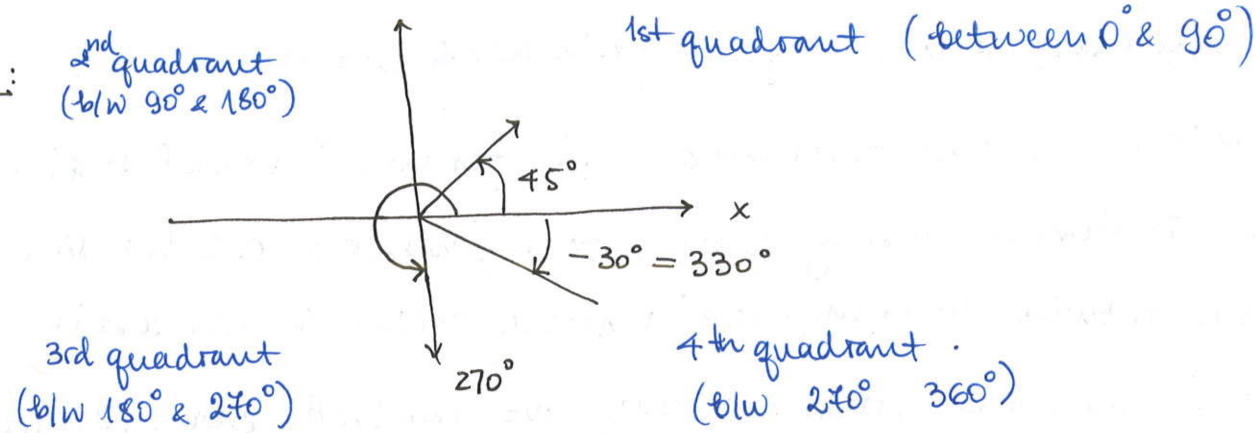


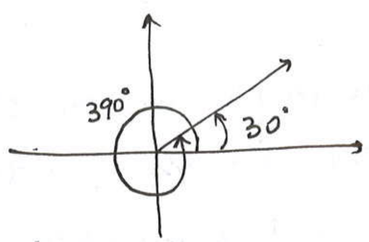
are quadrantal.

Ex:

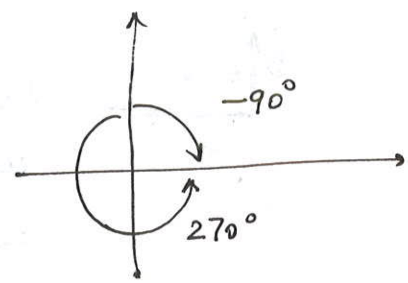


We call angles <u>co-terminal</u> if they have the same initial & the same terminal side.

Ex:



&



the angles above with measure 30° & 390° above are coterminal

the angles with measure −90° & 270° are co-terminal.

Thm: The angles α & β (in standard position) are coterminal if and only if

$$\boxed{\text{measure of } \beta = \text{measure of } \alpha + k\cdot 360^\circ \quad , k\in\mathbb{Z}.}$$

Ex: $m(\alpha) = 30^\circ$ here $m(\alpha)$ is the measure.

$m(\beta) = 390^\circ$

$\Rightarrow m(\beta) = m(\alpha) + (+1)\,360^\circ$

$360^\circ + 30^\circ = 390^\circ$

- $m(\alpha) = -90^\circ$ $\Rightarrow m(\beta) = m(\alpha) + (-1).360$

  $m(\beta) = 270^\circ$ $270 = -90^\circ + (-1)\,360^\circ.$

Ex: Are the angles α & β coterminal?

$m(\alpha) = 190^\circ$

$m(\beta) = -170^\circ$.

We know that if they are co-terminal: there is a $k \in \mathbb{Z}$ such that:

$$m(\beta) = m(\alpha) + k.360^\circ$$

$\Rightarrow -170^\circ = 190^\circ + k360^\circ$

$\Rightarrow -170^\circ - 190^\circ = k360^\circ$

$-360^\circ = k.360^\circ$

$k = -1$.

Ex: Which quadrant does the angle lie in?

a) $230^\circ = m(\alpha)$

b) $-580^\circ = m(\beta)$

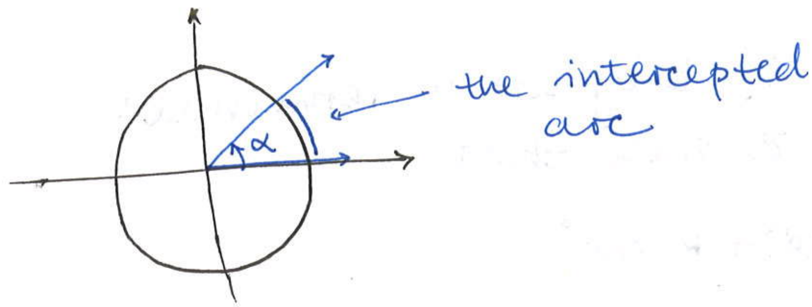
- Since $180^\circ < 230^\circ < 270^\circ \Rightarrow \alpha$ is in the third quadrant,

- since -580° is not between 0° & 360° so find a co-terminal angle to β, which is with measure betw. 0 & 360°. & place that in a quadrant.

$-580^\circ + 2.360 = 140^\circ$, $0^\circ < 140^\circ < 360^\circ$ (we determine what's the quadr. for this).

$90^\circ < 140^\circ < 180^\circ \Rightarrow \beta$ is in the second quadrant.

Def: Radian measure of an angle α in standard position is the length of the intercepted arc on the unit circle.



The unit circle = the circle of radius 1. $(r=1)$.
The circumference of the unit circle is. $2\pi r = 2\pi$.

Connect. between degree measure & radian measure:

- for degree measure we saw one complete revolution of the terminal side (ie. a whole circle) is $360°$
- for radian measure this means that the length of the intercepted arc is 2π = the circumference of the unit circle.

$$\Rightarrow \quad \boxed{2\pi = 360° \Rightarrow \pi = 180°}$$

or we sometimes write $2\pi \text{ rad} = 360°$.

$$\Rightarrow \text{1 degree} = 1 \text{ deg} \cdot \frac{\pi \text{ rad}}{180 \text{ deg}} = \frac{\pi}{180} \text{ rad}.$$

$$\Rightarrow 1 \text{ rad} = 1 \text{ rad} \cdot \frac{180 \text{ deg}}{\pi \text{ rad}} = \frac{180}{\pi} \text{ deg}$$

<u>Converting</u> from degrees to radians

a) $270^\circ \Rightarrow 270^\circ = \frac{270\pi}{180} = \frac{3\pi}{2}.$

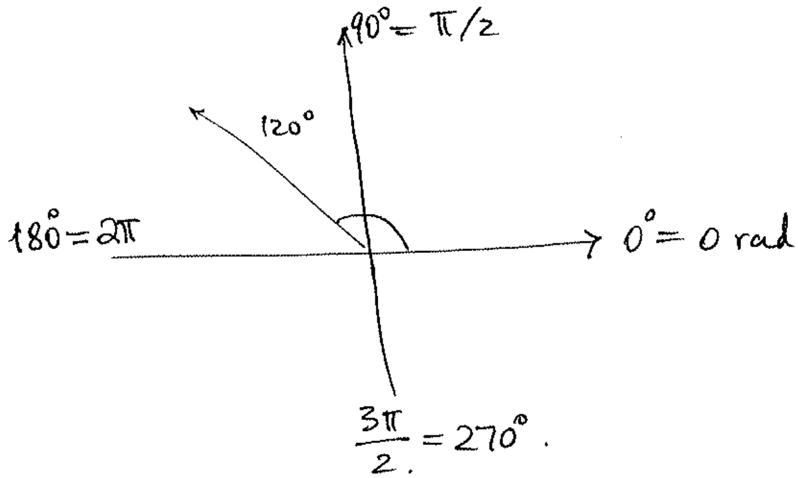
b) $-150^\circ \Rightarrow -150^\circ = \frac{-\cancel{150}^{5}\pi}{\cancel{180}_{6}} = -\frac{5}{6}\pi = \frac{1}{6}\pi$

<u>Converting</u> from radians to degrees.

a) $\frac{7}{6}\pi = \frac{7\pi}{\cancel{6}} \cdot \frac{\cancel{180}^{30}}{\pi} = 210^\circ.$

<u>Note</u>: Finding coterminal angles using radian measure: (same way as with angle measure, but instead of $k.360^\circ$ we add $k.2\pi$).

<u>Ex</u>: What quadrant is $\frac{2\pi}{3}$ is?



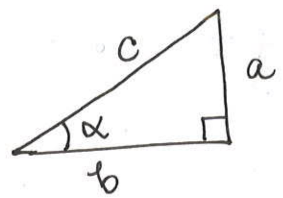
- $\frac{2\pi}{3} = \frac{2\pi}{3} \cdot \frac{180^\circ}{\pi} = 120^\circ$ & place in 2nd quadrant

- or just note that $\frac{\pi}{2} < \frac{2\pi}{3} < 2\pi \Rightarrow$ 2nd quadrant.

§5.2

(Sine and Cosine Functions).

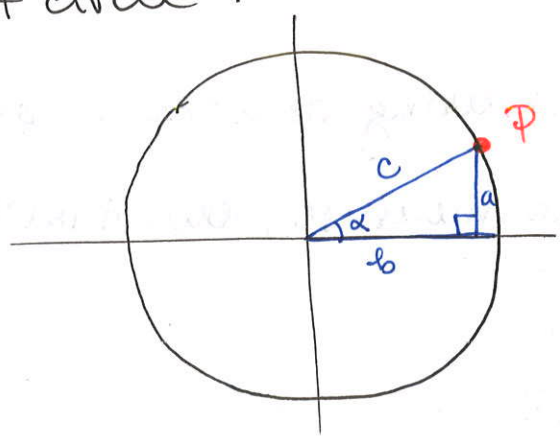
<u>Def</u>: Consider the right triangle:



the sine of $\alpha := \sin(\alpha) = \frac{a}{c}$

the cosine of $\alpha = \cos(\alpha) = \frac{b}{c}$.

Alternatively: consider this triangle inside the unit circle:



here $c = 1 =$ radius

$\Rightarrow \sin(\alpha) = a$

$\cos(\alpha) = b$

the point P has coordinates $(b, a) = (\cos\alpha, \sin\alpha)$.

The <u>alternative def.</u> for sin & cos is:

If α is an angle in standard position & (x,y) the point of intersection of the terminal side and the unit circle, then $x = \cos\alpha$, $y = \sin\alpha$.

- sin and cos in the quadrants:

2nd quadrant	1st Quadrant
$\sin\alpha > 0$ $\cos\alpha < 0$.	$\sin\alpha > 0$ $\cos\alpha > 0$.
3rd quadrant	**4th Quadrant**
$\sin\alpha < 0$ $\cos\alpha < 0$	$\sin\alpha < 0$ $\cos\alpha > 0$.

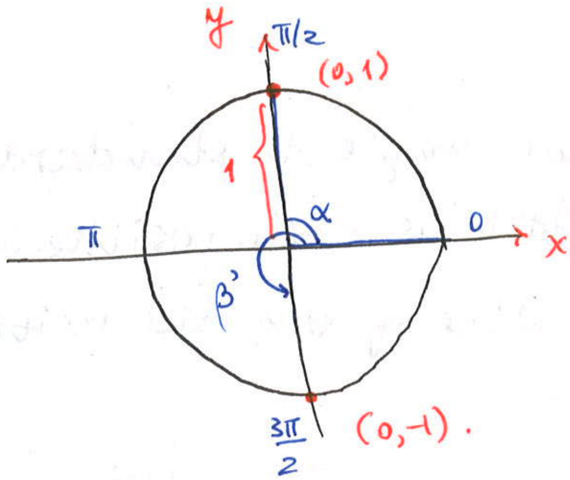
- Computing sin & cos at angles with measure a multiple of $\pi/2$ ie. $90°$. (look at the unit circle!)

$$\sin(90°) = \sin\frac{\pi}{2} = 1$$

$$\cos(90°) = \cos\frac{\pi}{2} = 0$$

$$\sin\left(-\frac{5\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) = -1$$

$$\cos\left(-\frac{5\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = 0.$$



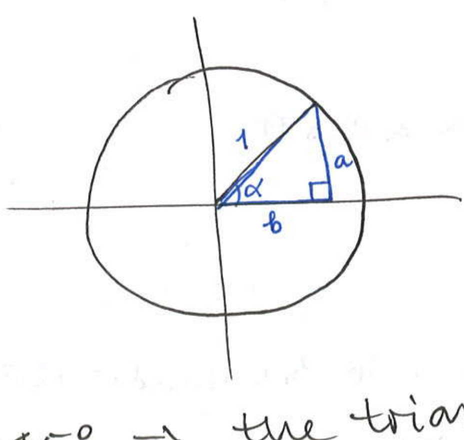
$\alpha = 90° = \pi/2$

$\beta = -\frac{5\pi}{2}$

$\beta' = \frac{3\pi}{2} = \beta + 2(2\pi)$

- for multiples of $30°$ & $45°$

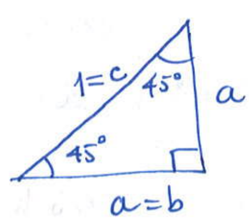
Recall the initial def of $\sin\alpha$ & $\cos\alpha$



$\sin\alpha = a$

$\cos\alpha = b$

if $\underline{\alpha = 45°} \Rightarrow$ the triangle is

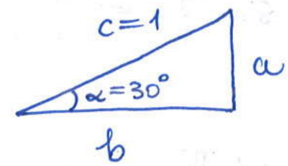


we know $a^2 + b^2 = c^2 \Rightarrow 2a^2 = 1 \Rightarrow a = \pm\sqrt{\frac{1}{2}} \Rightarrow a = \pm\frac{\sqrt{2}}{2}.$

& since we are in the 1st quadrant $\sin\alpha, \cos\alpha > 0$.

$\Rightarrow \sin\alpha = \frac{\sqrt{2}}{2} = \cos\alpha$.

if $\underline{\alpha = 30^\circ}$ $\Rightarrow$ the triangle is:



but $a = \frac{1}{2}c = \frac{1}{2}$

& $a^2 + b^2 = c^2 \Rightarrow \left(\frac{1}{2}\right)^2 + b^2 = 1^2 \Rightarrow b^2 = \frac{3}{4} \Rightarrow b = \pm\frac{\sqrt{3}}{2}$

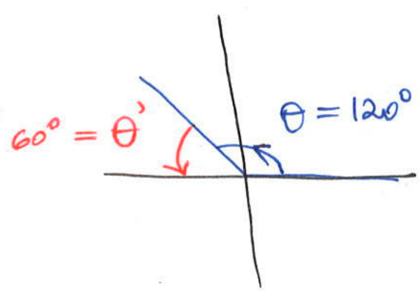
& since we are in the first quadrant again:

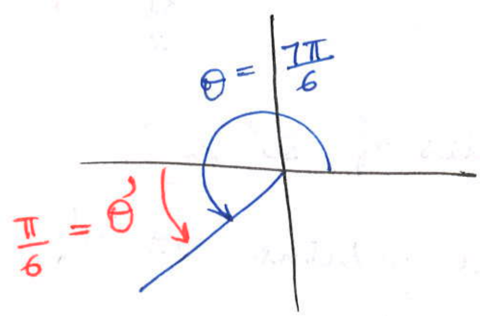
$\Rightarrow \sin\alpha = \frac{1}{2}$; $\cos\alpha = \frac{\sqrt{3}}{2}$.

Reference angles

Let θ be a non-quadrantal angle in standard position. The the reference angle for θ is a the positive acute angle θ', formed by the terminal side of θ & the positive or negative x-axis.

Ex





We can use this to compute $\sin\theta$ & $\cos\theta$:

Ex: $\sin\frac{7\pi}{6} = -\left(\sin\frac{\pi}{6}\right) = -\left(\frac{1}{2}\right)$

because the terminal side of $\frac{7\pi}{6}$ is in 3rd quadrant there $\sin\theta, \cos\theta < 0$. Similarly:

$\cos\frac{7\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$.

Ex: $\sin\theta$, for $\theta = 120^\circ = \frac{2\pi}{3}$

$$\sin 120^\circ = \sin\frac{2\pi}{3} = \sin\theta' = \sin 60^\circ = \frac{\sqrt{3}}{2}.$$

note that $\frac{2\pi}{3}$ is in the 2nd quadrant but there $\sin\theta > 0$.

$$\cos 120^\circ = \cos\frac{2\pi}{3} = -\cos\theta' = -\cos 60^\circ = -\frac{1}{2}$$

because in the second quadrant $\cos\theta < 0$.