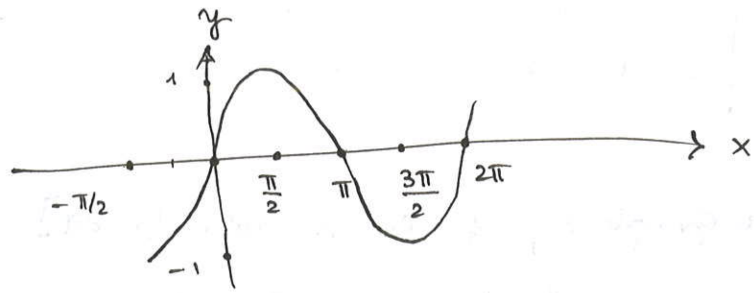
Chapter 5.3

(Graphs of Sin x & cos x).

1. Graph of $y = \sin x$.

x	0	$\pi/2$	π	$3\pi/2$	2π
y	0	1	0	-1	0



The graph is called a sine wave, sinusoidal wave or a sinusoid.

Observe: The graph of sin x looks the same ~~fo~~ in the intervals $[-2\pi, 0]$, $[0, 2\pi]$, $[2\pi, 4\pi]$, $[4\pi, 6\pi]$, ... This is normal since we observed that the sine value of an angle and its co-terminal are the same i.e. $\sin\alpha = \sin(\alpha + 2\pi)$. Thus every 2π we get the same picture.

We call functions like this periodic.

Def: Let $f(x) = y$ be a function and a - a constant. $a \neq 0$. such that $f(x+a) = f(x)$, for every x in the domain of $f(x)$. Then f is called a periodic function & the smallest such a is called ~~a~~ the period of f.

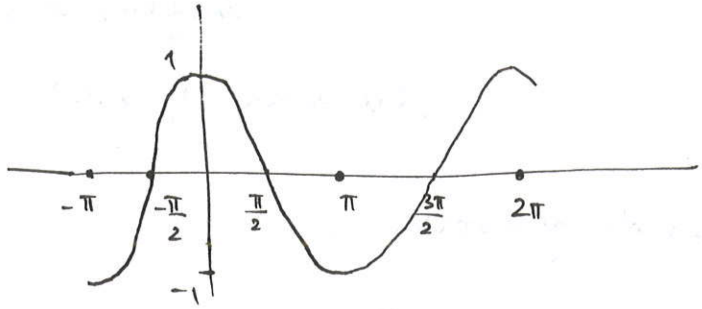
Note: For $\sin(x)$ the period is 2π.

Def: The graph of $y = \sin x$ over $[0, 2\pi]$ is called a fundamental cycle

The graph of $y = \sin x$ over any interval of length 2π is called a cycle.

2. Graph of $y = \cos x$.

x	0	$\pi/2$	π	$3\pi/2$	2π
y	1	0	-1	0	1.



. The graph of $y = \cos x$ over $[0, 2\pi]$ is called the <u>fundamental</u> cycle.

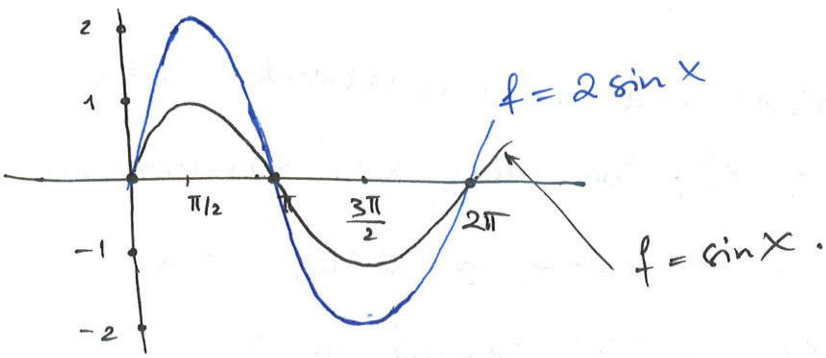
. The <u>period</u> here is also 2π.

3. Transformations of $\sin x$ & $\cos x$.

Sketch the graph of $f = 2\sin x$ on $[-2\pi, 2\pi]$. ie : $\boxed{af(x)}$

x	0	$\pi/2$	π	$3\pi/2$	2π
y	0	2	0	-2	0

(note the function's graph is the same on $[-2\pi, 0]$ as on $[0, 2\pi]$)



<u>Note</u>: the zeroes (x-interc.) of the function remain the same. The "hight" of the wave changes.

<u>Def</u>: The "hight" of the sine wave is measured by the amplitude.

The amplitude of a function $f(x)$ is $\frac{1}{2}\left|\begin{matrix}\text{max } y\text{-coord.} \\ \text{of } f(x)\end{matrix} - \begin{matrix}\text{min. } y\text{-coord} \\ \text{of } f(x)\end{matrix}\right|$

<u>Ex</u>: for $f(x) = \sin x$

max y-value is 1
min. y-value is -1

$\Big\}$ amplitude $= \frac{1}{2}\left|1 - (-1)\right| = 1$.

Ex: $f(x) = 2\sin x$

max y-value is 2
min y-value is -2 $\Big\}$ amplitude $= \frac{1}{2}\left|2-(-2)\right| = 2.$

Theorem: The amplitude of $f(x) = a\sin x$ or $f(x) = a\cos(x)$ is $|a|$.

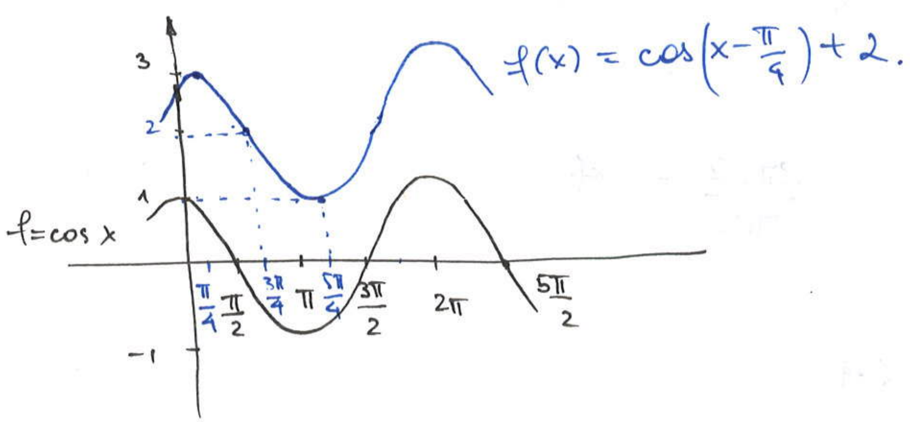
Def: **Phase shift:** of the graph of $f(x) = \sin(x-C)$ or $f(x) = \cos(x-C)$ is C. This is the horizontal shift! (translation).

The horizontal and vertical translations work exactly the same way as in every other function:

Ex: Sketch $y = \cos\left(x - \frac{\pi}{4}\right) + 2.$ is of the form: $\boxed{f(x-h)+k}$

horizontal $\frac{\pi}{4}$ to the right (= phase shift).

vertical: 2 up



Ex: Sketch $f(x) = \sin(2x)$ ie. of the form $\boxed{f(dx)}$, $d \neq 0$.

Note: this d changes the period of the function.

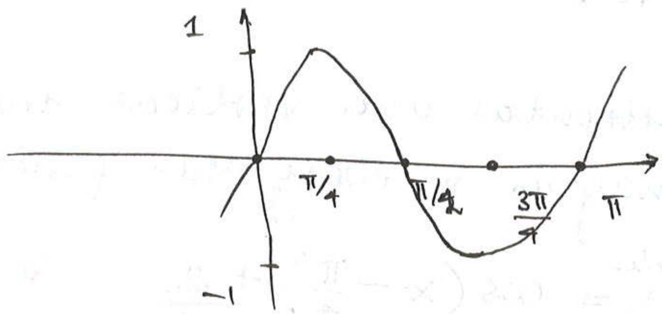
first compare the fundamental cycles:

for $f(x) = \sin x$ the fundamental cycle is $0 \leq x \leq 2\pi$

for $f(x) = \sin(2x)$ the fundamental cycle is $0 \leq 2x \leq 2\pi$ ie. $0 \leq x \leq \pi$

$\Rightarrow$ the period is π.

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$\sin(2x)$	0	1	0	-1	0



Theorem: The period of $y = \sin(Bx)$ or $y = \cos(Bx)$ for $B > 0$ is (given by) $\frac{2\pi}{B}$.

Ex: Determine the period of $f = \cos(\frac{\pi}{2}x)$.

Here $B = \frac{\pi}{2}$

the period is $\dfrac{2\pi}{\frac{\pi}{2}} = \dfrac{2\pi \cdot 2}{\pi} = 4.$

$\Rightarrow$ the cycle is $0 \leq x \leq 4$

it starts at $(0,1)$ & is completed at $(4,1)$.

Chapter 5.4

Other trig functions & their graphs

Def: Let α be an angle in standard position & $(x,y) = (\cos\alpha, \sin\alpha)$ (the point where the terminal side intersects the unit circle). Then we define the following functions:

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{y}{x} \quad \text{- tangent}$$

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} = \frac{x}{y} = \frac{1}{\tan\alpha} \quad \text{- cotangent}$$

$$\sec\alpha = \frac{1}{\cos\alpha} = \frac{1}{y} \quad \text{- secant}$$

$$\csc\alpha = \frac{1}{\sin\alpha} = \frac{1}{x} \quad \text{- cosecant}$$

Note: these are only defined when the denominator isn't 0!

Computing the domains:

- The domains of $\tan\alpha$ & $\sec\alpha$ are the set of angles for which $\cos\alpha \neq 0$. The only pairs on the unit circle for which the x-coordinate is 0 (ie. $\cos\alpha = 0$) are $(0,1)$ & $(0,-1)$, that is for angles $\pi/2, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$ ie.

$$\cos\alpha = 0 \quad \text{for} \quad \alpha \in \left\{\alpha \mid \alpha = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\}$$

that is: $\cos\alpha = 0$ if α is of the form $\frac{\pi}{2} + k\pi$, where k is an integer.

- similarly for the domains of $\cot\alpha$ & $\csc\alpha$ we would include all angles α for which $\sin\alpha \neq 0$.

These are angles of the form α, where $\alpha \neq k\pi$, k - integer.

(because $\underset{\sin(0.\pi)\text{ ie. } k=0}{\sin(0)} = 0$, $\underset{\sin(1.\pi)\text{ ie } k=1}{\sin(\pi)} = 0$, $\underset{\sin(2\pi)\text{ ie. } k=2.}{\sin(2\pi)} = 0$, ...)

Evaluating trig. functions (this is easy when we know $\sin\alpha$ & $\cos\alpha$).

Ex: $\tan\frac{\pi}{4} = \frac{\sin\pi/4}{\cos\pi/4} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1.$

$$\sec\frac{\pi}{4} = \frac{1}{\cos\pi/4} = \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Graphs of trig functions

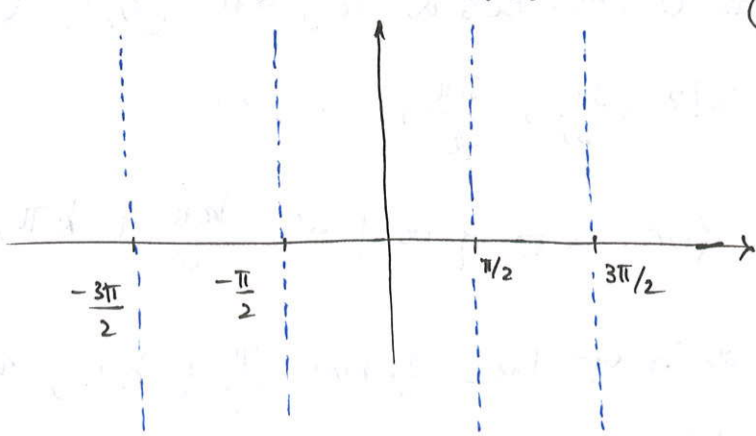
- Graph of $y = \tan x = \frac{\sin x}{\cos x}$ (we will do here an analysis similar to the one we did when sketching rational functions).

  - when $\cos\alpha = 0$ the function is not defined.

  we saw above that this happens for angles $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

  ie. of the form $\frac{\pi}{2} + k\pi$ for k - an integer.

  At these values we have vertical asymptotes:



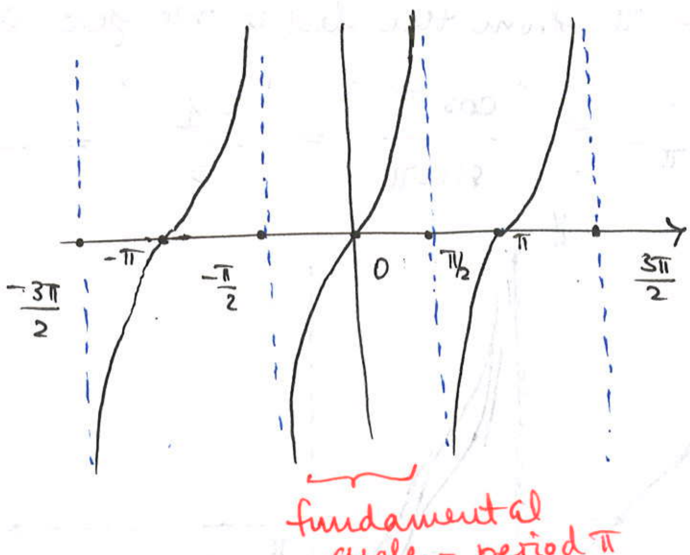
as "$\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty$" this means, when we go infinitely close to $\frac{\pi}{2}$ from the left the function becomes infinitely large, but positive, because $\sin x > 0$ & $\cos x \to 0^+$ (very small positive)

$$\| $$

"$\lim_{x \to \frac{\pi}{2}^-} \frac{\sin x}{\cos x}$"

we are getting something of the form "$\lim_{x \to a} \frac{b^{>0}}{0} = \infty$"

similarly when x appoaches $-\frac{\pi}{2}$ from the right we get $-\infty$ because $\sin x < 0$ & $\cos x \to 0^+$. (very small positive).



also note that the x-intercept corresponds to $\sin\alpha = 0$

because then $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{0}{\cos\alpha} = 0$.

this happens at $0, \pm\pi, \pm 2\pi, -3\pi, \dots$

The function is periodic (period π). That is it looks the same on $[-\pi/2, \pi/2]$, $[-\frac{3\pi}{2}, -\pi/2]$, $[\frac{\pi}{2}, \frac{3\pi}{2}]$, etc.

the range is $(-\infty, \infty)$.

we can do a similar thing for $\cot x$.

- Graph $\cot x = \dfrac{\cos x}{\sin x}$

This function is not defined when $\sin \alpha = 0$ ie. $\alpha = 0, \pm\pi, \pm 2\pi, \ldots$

The function has vert. asymptotes at these points $x = 0, \pm\pi, \pm 2\pi$.

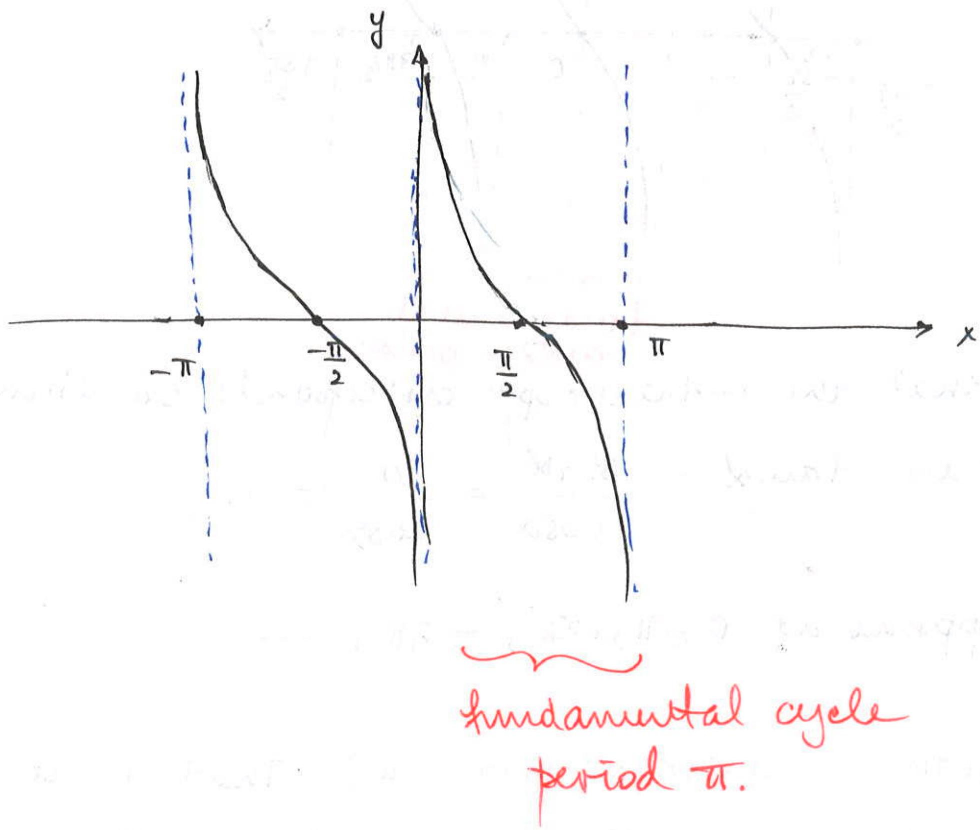
The function has x-intercept when $\cos\alpha = 0$ ie. $\alpha = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

when we approach 0 from the right we get a $+\infty$

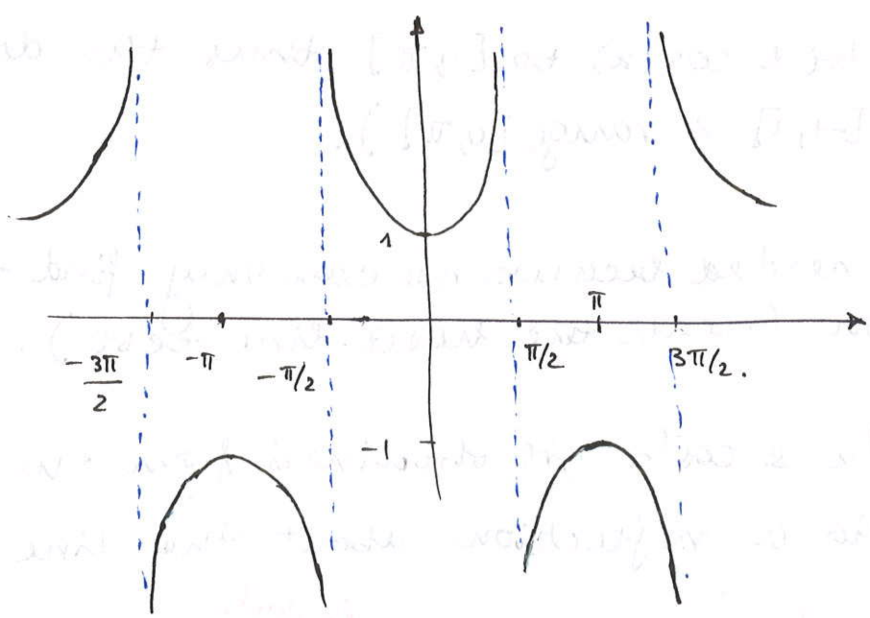
because then $\displaystyle\cot_{a\to 0^+} a = \text{"}\frac{\cos 0}{\sin 0}\text{"} = \text{"}\frac{1}{0}\text{"} = \infty$

when we approach π from the left we get a $-\infty$

because then $\displaystyle\cot_{a\to \pi^-} a = \text{"}\frac{\cos \pi}{\sin \pi}\text{"} = \text{"}\frac{-1}{0}\text{"} = -\infty$.



- Graph of sec(x) = 1/cos x
  - there are asymptotes when cos x = 0.
  - when cos x is small 1/cos x is large & vice versa ⇒ ±1 are min & max for sec(x). at $x = 0, \pm\pi, \pm 2\pi$
  - there are no x-intercepts
  - y intercepts for $x = 0 \Rightarrow \sec(0) = 1$.
  - for $\cos x > 0$, $\cos x \to 0 \Rightarrow 1/\cos x \to \infty$
  - for $\cos x < 0$, $\cos x \to 0 \to 1/\cos x \to -\infty$



fundamental cycle - period 2π

- Graph of csc(x) = 1/sin x

it looks like the graph of sec(x) shifted to the left ie.

asymptotes are now at $0, \pm\pi, \pm 2\pi, \dots$

min & max of the funct. at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \dots$

Chapter 5.5

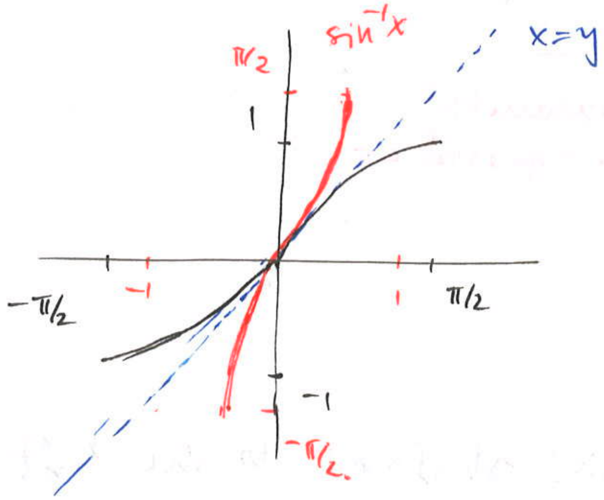
(Inverse trig. functions)

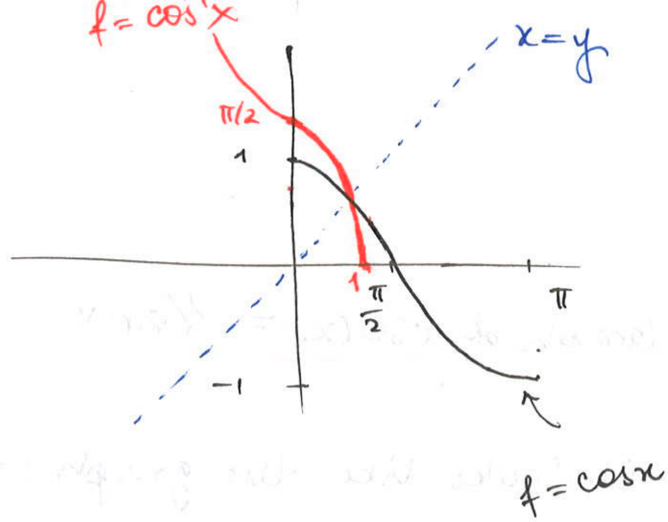
Def: The inverse of the sine function is denoted by $\sin^{-1} x$ or $\arcsin x$ (read sine inverse or arc-sine of x)
(note here we restrict $\sin(x)$ to $[-\pi/2, \pi/2]$. Thus the domain of $\sin^{-1} x$ is $[-1, 1]$ & range $[-\pi/2, \pi/2]$).

Def: The inverse of the cosine function is denoted by $\cos^{-1} x$ or $\arccos x$ (read cosine inverse or arccosine of x).
(here we restrict $\cos(x)$ to $[0, \pi]$, thus the domain of $\cos^{-1} x$ is $[-1, 1]$ & range $[0, \pi]$).

The restriction is needed because we can only find the inverse of a 1-1 function. (recall the horiz. line test).

The graphs of $\sin^{-1} x$ & $\cos^{-1} x$ are obtained from the graphs of $\sin x$ & $\cos x$ via a reflection about the line $x = y$.





$\sin^{-1}(\sin x) = x$ & $\cos^{-1}(\cos x) = x$

$\sin(\sin^{-1} x) = x$ & $\cos(\cos^{-1} x) = x$

Evaluating inverse functions:

Ex: Find the exact value of:

a) $\sin^{-1}(1/2) = \arcsin(1/2) = ?$

Let $\sin^{-1}\left(\frac{1}{2}\right) = \alpha \Rightarrow$ take sin on both sides:

$$\sin\left(\sin^{-1}\left(\frac{1}{2}\right)\right) = \sin\alpha$$

$$\frac{1}{2} = \sin\alpha$$

$\Rightarrow \alpha = \frac{\pi}{6}$ (choose an angle in the range $[-\pi/2, \pi/2]$.)

b) $\arccos(\sqrt{2}/2) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = ?$

let $\arccos\left(\frac{\sqrt{2}}{2}\right) = \beta$, take cos on both sides:

$$\cos\left(\arccos\frac{\sqrt{2}}{2}\right) = \cos\beta$$

$$\frac{\sqrt{2}}{2} = \cos\beta$$

$\Rightarrow \beta = \frac{\pi}{4}$ (angle in the range $[0, \pi]$.).

The inverses of tangen, cotangent, secant & cosecant are obtained in a similar fasion. First restrict them to the (fundamental) cycle or any other cycle to obtain a 1-1 funct. The graph of the inverse is obtained as usual from the graph of ~~of~~ the original after reflection about the $x = y$ line.

Identities for the inverse functions:

$$\csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right) \qquad |x| \geq 1.$$

$$\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right) \qquad |x| \geq 1.$$

$$\cot^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right) & \text{for } x > 0. \\ \tan^{-1}\left(\frac{1}{x}\right) + \pi & \text{for } x < 0. \\ \pi/2 & \text{for } x = 0. \end{cases}$$

$$\cot^{-1} x = \pi/2 - \tan^{-1} x.$$

Evaluating inverse trigs:

Ex: $\tan^{-1}(1) = \alpha$

$\tan\alpha = 1$

$\alpha = \underline{\underline{\pi/4}}.$

Ex: $\cot^{-1}\left(-\frac{\sqrt{3}}{3}\right)$ use the third identity.

$$\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right) + \pi \quad , \text{ for } x < 0 \qquad \left(\text{here } x = -\frac{\sqrt{3}}{3}\right).$$

$$\cot^{-1}\left(-\frac{\sqrt{3}}{3}\right) = \tan^{-1}\left(-\frac{3}{\sqrt{3}}\right) + \pi$$

$$\tan^{-1}\left(-\frac{3}{\sqrt{3}} = -\sqrt{3}\right) = \alpha \Rightarrow \quad \tan\alpha = -\sqrt{3} = -\frac{\sin\alpha}{\cos\alpha} = -\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

$$\Rightarrow \alpha = -\frac{\pi}{3} = -60^\circ$$

$$\Rightarrow \cot^{-1}\left(-\frac{\sqrt{3}}{3}\right) = -\frac{\pi}{3} + \pi = \underline{\underline{\frac{2}{3}\pi}}.$$

Ex: $\arcsin(\underbrace{\cos\frac{\pi}{6}}) = \arcsin\left(\underbrace{\frac{\sqrt{3}}{2}}\right) = \alpha$

$\Rightarrow \sin\alpha = \frac{\sqrt{3}}{2} \Rightarrow \alpha = \frac{\pi}{3}.$

Finding the (general) inverse of a general (sine) funct. (trig.)

Ex: $y = 3\sin(2x) + 5. \qquad -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}.$

. interch. x & y. : $x = 3\sin 2y + 5$

. solve for y : $x - 5 = 3\sin 2y$

$\frac{x-5}{3} = \sin 2y$ (take arc sin).

$\arcsin\left(\frac{x-5}{3}\right) = 2y$

$y = \frac{1}{2}\arcsin\left(\frac{x-5}{3}\right). = f^{-1}(x).$

the range of f is $[2,8]$ $\Rightarrow$ domain of f^{-1} is $[2,8]$.

(b/c $-1 \leq \sin(2x) \leq 1$).