# Trig Identities
## § 6.1 & 6.2

**Recall:**

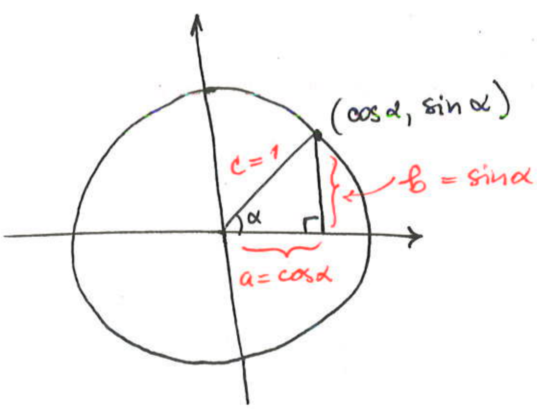
$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ $\quad \sec\alpha = \frac{1}{\cos\alpha}$ & $\cos\alpha = \frac{1}{\sec\alpha}$

$\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$ $\quad \csc\alpha = \frac{1}{\sin\alpha}$ & $\sin\alpha = \frac{1}{\csc\alpha}$

$\tan\alpha = \frac{1}{\cot\alpha}$ & $\cot\alpha = \frac{1}{\tan\alpha}$

Because of the way $\sin x$ & $\cos x$ are defined we can obtain one more identity: the pythagorean identity:

**Recall:**




we know that

$a^2 + b^2 = c^2$ in this right $\Delta$.

$\Rightarrow \boxed{\cos^2\alpha + \sin^2\alpha = 1}$

Use this identity to obtain two more:

$\sin^2\alpha + \cos^2\alpha = 1$ $\quad$ (divide by $\cos^2\alpha$).

$$\frac{\sin^2\alpha}{\cos^2\alpha} + \frac{\cos^2\alpha}{\cos^2\alpha} = \frac{1}{\cos^2\alpha}$$

$$(\tan\alpha)^2 + 1 = \frac{1}{\cos^2\alpha} = \sec^2\alpha. \quad \Rightarrow \boxed{\tan^2\alpha + 1 = \sec^2\alpha}$$

Similarly:

$$\sin^2\alpha + \cos^2\alpha = 1 \quad (\text{divide by } \sin^2\alpha)$$

$$\frac{\sin^2\alpha}{\sin^2\alpha} + \frac{\cos^2\alpha}{\sin^2\alpha} = \frac{1}{\sin^2\alpha}.$$

$$1 + \left(\frac{\cos\alpha}{\sin\alpha}\right)^2 = \left(\frac{1}{\sin\alpha}\right)^2 \Rightarrow$$

$$\boxed{1 + \cot^2\alpha = \csc^2\alpha.}$$

Ex: Simplify:

$$\sin x + \cot x \cos x =$$

step 1 rewrite everything sin x & cos x only:

$$\sin x + \frac{\cos x}{\sin x} \cdot \cos x =$$

$$\underbrace{\frac{\overset{\sin x}{\sin x}}{1} + \frac{\cos^2 x}{\sin x}}_{\sin x} =$$

step 2 try to sum up.

$$\frac{\sin^2 x + \cos^2 x}{\sin x} =$$

step 3 use identities

$$= \frac{1}{\sin x} = \csc x.$$

Ex: Write sin(x) in terms of cot(x).

$$1 + \cot^2 x = \frac{1}{\sin^2 x}$$

$$\sin^2 x = \frac{1}{1+\cot^2 x}$$

$$\sin x = \pm \frac{1}{\sqrt{1+\cot^2 x}}$$

Ex: Let $\alpha$ be in the IV quadrant & $\tan\alpha = -\frac{2}{3}$.

Find $\sin\alpha$, $\cos\alpha$, $\cot\alpha$, $\sec\alpha$, $\csc\alpha$.

- $\tan^2\alpha + 1 = \sec^2\alpha$

$$\Rightarrow \sec^2\alpha = \frac{4}{9} + 1 = \frac{13}{9}$$

$$\sec\alpha = \pm\frac{\sqrt{13}}{\sqrt{9}} = \pm\frac{\sqrt{13}}{3}$$

but $\sec\alpha = \frac{1}{\cos\alpha}$ & in the IV quadrant $\cos\alpha > 0$ (& hence $1/\cos\alpha > 0$)

$$\Rightarrow \sec\alpha = +\frac{\sqrt{13}}{3}$$

& $\cos\alpha = \frac{3}{\sqrt{13}} = \frac{3\sqrt{13}}{13}$

- $\cot\alpha = 1/\tan\alpha = -\frac{3}{2}$
- $\sin\alpha = \pm\sqrt{1-\cos^2\alpha}$ (in IV quadrant $\sin\alpha < 0$) so:

$$\sin\alpha = -\sqrt{1-\frac{9}{13}} = -\sqrt{\frac{4}{13}} = -\frac{2}{\sqrt{13}} = -\frac{2\sqrt{13}}{13}.$$

## Even & Odd identities

Even functions: $\cos x$ & $\frac{1}{\cos x} = \sec x$ $\qquad f(-x) = f(x)$

Odd functions: $\underbrace{\sin x, \tan x, \cot x, \csc x.}_{\text{these are odd b/c } \sin x \text{ is odd \& } \cos x \text{ is even.}}$ $\qquad f(-x) = -f(x)$

Ex
Simplify

$$\frac{1}{1+\underbrace{\cos(-x)}_{=\cos x}} + \frac{1}{1-\cos x} = \underbrace{\frac{1}{1+\cos x}\cdot\frac{1-\cos x}{1-\cos x} + \frac{1}{1-\cos x}\cdot\frac{1+\cos x}{1+\cos x}} = \frac{1-\cos x + 1+\cos x}{1-\cos^2 x}$$

$$= \frac{2}{\sin^2 x} = 2\csc^2 x.$$

Proving that something is an identity:

- sometimes work with only one side (until you get the other)
- sometimes try to manipulate both.
- helps to write everything in terms of $\sin x$ & $\cos x$ only.

(these are very tricky simplification problems).

Ex: Show that the following is an identity.

- $\sec^2 x = \underbrace{1 + \sec x \sin x \cdot \tan x}$.

manipulate this side - write ~~everything~~ everything as $\sin x$ & $\cos x$ only

$$= 1 + \frac{1}{\cos x} \cdot \sin x \cdot \frac{\sin x}{\cos x}$$

$$= 1 + \tan^2 x$$ ✓ (one of the pythagorean identities).

- $\cot^2 x = \underbrace{\dfrac{\csc x - \sin x}{\sin x}}$

manipulate this side - write as two fractions

$$= \frac{\frac{1}{\sin x}}{\sin x} - \frac{\sin x}{\sin x} = \frac{1}{\sin^2 x} - 1 = \csc^2 x - 1.$$ ✓

because $\cot^2 x + 1 = \csc^2 x$.

- $2\tan^2 x = \underbrace{\dfrac{1}{\csc x - 1}^{\csc x + 1} - \dfrac{1}{\csc x + 1}^{\csc x - 1}} = \dfrac{\csc x + 1 - \csc x + 1}{\csc^2 x - 1} = \dfrac{2}{\cot^2 x}$

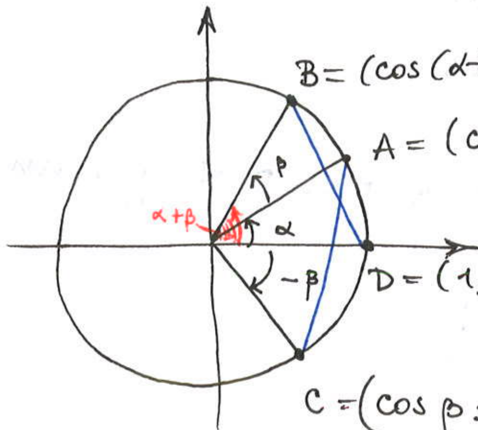
do the subtract.

$\cancel{2}\tan^2 x = \dfrac{\cancel{2}}{\cot^2 x}$ b/c $\tan x = \dfrac{1}{\cot x}$. ✓

# Chapter 6.3

(Sum & difference identities).

Consider the unit circle; angles $\alpha$ & $\beta$ as below:




Observe: $\measuredangle BOD = \measuredangle AOC \Rightarrow BD = AC$.

$$BD = \sqrt{(\cos(\alpha+\beta)-1)^2 + (\sin(\alpha+\beta)-0)^2}$$

$$= \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2} \text{ — the distance formula}$$

$$=$$

$$AC = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - (-\sin\beta))^2}$$

simplify $\left((\cos(\alpha+\beta)-1)^2 + (\sin(\alpha+\beta)-0)^2\right)^{1/2} =$

$$\left((\cos\alpha - \cos\beta)^2 + (\sin\alpha + \sin\beta)^2\right)^{1/2}.$$

(square the binomials)

$$(\cos(\alpha+\beta)-1)^2 + (\sin(\alpha+\beta))^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha + \sin\beta)^2$$

$$\boxed{(\cos(\alpha+\beta))^2} - 2\cos(\alpha+\beta) + 1 + \boxed{(\sin(\alpha+\beta))^2} =$$

$$\boxed{\cos^2\alpha} + \cos^2\beta - 2\cos\alpha\cos\beta + \boxed{\sin^2\alpha} + \sin^2\beta + 2\sin\alpha\sin\beta.$$

(use $\sin^2\alpha + \cos^2\alpha = 1$

$\sin^2\beta + \cos^2\beta = 1$

$\sin^2(\alpha+\beta) + \cos^2(\alpha+\beta) = 1$ )

$$2 - 2\cos(\alpha+\beta) = 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta$$

$$\Rightarrow \boxed{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

for computing $\cos(\alpha-\beta)$ we just replace $\beta$ with $(-\beta)$ in the formula above and use the fact that $\cos x$ is even & $\sin x$ is odd:

$$\boxed{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.}$$

Cofunction identities: (these follow from the ones we just showed.

| | |
|---|---|
| $\sin\left(\frac{\pi}{2} - u\right) = \cos u$ | $\cos\left(\frac{\pi}{2} - u\right) = \sin u$ |
| $\tan\left(\frac{\pi}{2} - u\right) = \cot u$ | $\cot\left(\frac{\pi}{2} - u\right) = \tan u.$ |
| $\sec\left(\frac{\pi}{2} - u\right) = \csc u.$ | $\csc\left(\frac{\pi}{2} - u\right) = \sec u.$ |

Proof (for the ~~1st~~ 2nd one)

$$\cos\left(\frac{\pi}{2} - u\right) = \underbrace{\cos\frac{\pi}{2}}_{0} \cdot \cos u + \underbrace{\sin\frac{\pi}{2}}_{1} \sin u$$

$$= 0 \cdot \cos u + \sin u = \sin u.$$

if $u = \frac{\pi}{2} - w \Rightarrow \sin u = \sin\left(\frac{\pi}{2} - w\right) = \cos\left(\frac{\pi}{2} - \frac{\pi}{2} + w\right) = \cos w$

$$\Rightarrow \sin\left(\frac{\pi}{2} - w\right) = \cos w.$$

Ex Find the value of $\sin(5\pi/12)$.

$$\sin\left(\frac{5\pi}{12}\right) = \cos\left(\frac{\pi}{2} - \frac{5\pi}{12}\right)$$

$$= \cos\left(\frac{\pi}{12}\right)$$

note that $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$

$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3} \cdot \cos\frac{\pi}{4} + \sin\frac{\pi}{3} \cdot \sin\frac{\pi}{4}.$$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2} + \sqrt{6}}{4}$$

$$\Rightarrow \sin\left(\frac{5\pi}{12}\right) = \frac{\sqrt{2} + \sqrt{6}}{4}.$$

Using the co-function identities we can obtain formulas for the sine of sum or difference of angles.

$$\sin(\alpha+\beta) = \cos\left(\frac{\pi}{2} - (\alpha+\beta)\right) = \cos\left(\left(\frac{\pi}{2} - \alpha\right) - \beta\right)$$

$$= \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta + \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta.$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

$$\boxed{\begin{aligned} \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \end{aligned}}$$

$$\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

(divide numerator & denominator by $1/\cos\alpha\cos\beta$)

$$= \frac{\frac{\sin\alpha\,\cancel{\cos\beta}}{\cos\alpha\,\cancel{\cos\beta}} + \frac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\frac{\cancel{\cos\alpha\cos\beta}}{\cancel{\cos\alpha\cos\beta}} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$\Rightarrow$ we get:

$$\boxed{\begin{aligned} \tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}. \\ \tan(\alpha-\beta) &= \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} \end{aligned}}$$

Use the sum & difference identities to simplify the following:

Ex

a) $\cos 47^\circ \cos 2^\circ + \sin 47^\circ \sin 2^\circ = \cos(47^\circ - 2^\circ) = \cos(45^\circ) = \frac{\sqrt{2}}{2}.$

b) $\sin t \cos 2t - \cos t \sin 2t = \sin(t - 2t) = \sin(-t) = -\sin t.$

Ex: Find the exact value of $\sin(\alpha+\beta)$ if $\sin\alpha = -3/5$. $\cos\beta = -1/3$.

(if $\alpha$ is in the 4th quadrant
$\beta$ is in the 3rd quadrant)

use: $\sin^2\alpha + \cos^2\alpha = 1$

$\sin^2\beta + \cos^2\beta = 1.$

to get:

- $\left(-\frac{3}{5}\right)^2 + \cos^2\alpha = 1 \Rightarrow \cos^2\alpha = \frac{16}{25} \Rightarrow \cos\alpha = \pm 4/5.$

because $\alpha$ is in the 4th quadrant $\cos\alpha > 0 \Rightarrow \cos\alpha = 4/5.$

- $\sin^2\beta + \left(-1/3\right)^2 = 1 \Rightarrow \sin^2\beta = 8/9 \Rightarrow \sin\beta = \pm\frac{2\sqrt{2}}{3}.$

because $\beta$ is in the 3rd quadrant $\sin\beta < 0 \Rightarrow \sin\beta = -\frac{2\sqrt{2}}{3}.$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$= \left(-\frac{3}{5}\right)\left(-\frac{1}{3}\right) + \frac{4}{5}\left(-\frac{2\sqrt{2}}{3}\right) = \frac{3}{15} - \frac{8\sqrt{2}}{15} = \frac{3-8\sqrt{2}}{15}.$$

## § 6.4 Double angle identities:

$$\boxed{\begin{aligned} \sin 2x &= 2\sin x\cos x \\ \cos 2x &= \cos^2 x - \sin^2 x \\ \cos 2x &= 2\cos^2 x - 1. \\ \cos 2x &= 1 - 2\sin^2 x \\ \tan 2x &= \frac{2\tan x}{1-\tan^2 x}. \end{aligned}}$$

Proof:

$$\sin 2x = \sin(x+x) = \sin x\cos x + \cos x\sin x = 2\sin x\cos x.$$

$$\cos 2x = \cos(x+x) = \cos x\cos x + \sin x\sin x = \cos^2 x - \sin^2 x$$

use the fact that $1 = \cos^2 x + \sin^2 x$ to obtain the ~~n~~ext two.

$$\tan(2x) = \tan(x+x) = \frac{\tan x + \tan x}{1 - \tan x\tan x} = \frac{2\tan x}{1-\tan^2 x}.$$

From here we can obtain the half angle identities:

for example: $2\cos^2 x - 1 = \cos 2x$

$$\cos^2 x = \frac{\cos 2x + 1}{2}$$

$$\Rightarrow \cos x = \pm\sqrt{\frac{\cos 2x+1}{2}}$$

$$\Rightarrow \boxed{\cos\frac{\alpha}{2} = \pm\sqrt{\frac{\cos\alpha+1}{2}}}$$

similarly:

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

$$\tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha} \qquad \tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha}.$$

Ex: Use the above identities to compute:

a) $\cos \pi/8 \Rightarrow \cos\frac{\pi/4}{2} = \sqrt{\frac{1+\cos\pi/4}{2}} = \sqrt{\frac{1+\sqrt{2}/2}{2}} = \frac{\sqrt{2+\sqrt{2}}}{2}$

b) $\sin(-22.5°) \Rightarrow \ldots$

1st quadrant

$\cos 150° = -\cos 30°$

c) $\sin(75°) \Rightarrow \sin(75°) = \sin\left(\frac{150°}{2}\right) = \pm\sqrt{\frac{1-\cos 150°}{2}} = +\sqrt{\frac{1+\cos 30°}{2}}$

$$= \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

d) $\tan(-15°) = \frac{\sin(-30°)}{1+\cos(-30°)} = \frac{-\sin 30°}{1+\cos 30°} = \frac{-1/2}{1+\frac{\sqrt{3}}{2}}$

$$= \frac{-\frac{1}{2}}{\frac{2+\sqrt{3}}{2}} = -(2+\sqrt{3}) = -2+\sqrt{3}$$

Verify the identities.

<u>Ex</u> $\tan\frac{x}{2} + \cot\frac{x}{2} = 2\csc x$.

manipulate the l.h.s (left hand side).

$$= \tan\frac{x}{2} + \frac{1}{\tan\frac{x}{2}} = \frac{\sin x}{1+\cos x} + \frac{1}{\frac{1-\cos x}{\sin x}}$$

$$= \frac{\overbrace{\sin x}^{1-\cos x}}{1+\cos x} + \frac{\overbrace{\sin x}^{1+\cos x}}{1-\cos x} = \frac{\sin x(1-\cos x) + \sin x(1+\cos x)}{1-\cos^2 x}$$

$$= \frac{2\sin x}{\sin^2 x} = \frac{2}{\sin x} = \underline{2\csc x}.$$