Question 6.1.78

Question 1 *Without computing the integral show that*

$$\frac{1}{2} \le \int_0^1 \sqrt{1-x^2}\ dx \le 1.$$

We know that if $m \le f(x) \le M$, for some constants m and M, then

$$m(a-b) \le \int_a^b f(x)\ dx \le M(a-b).$$

Note that on the interval $[0,1]$, $0 \le \sqrt{1-x^2} \le 1$, hence

$$0 \le \int_0^1 \sqrt{1-x^2}\ dx \le 1,$$

We have the upper bound that we want but not the lower bound. To get a better lower bound, we will think think of the integral as area, that is $\int_0^1 \sqrt{1-x^2}\ dx$ is the area A under the graph of $\sqrt{1-x^2}$ on the interval $[0,1]$. Note that we can write this area in the following way:

$$A = \int_0^{1/\sqrt{2}} \sqrt{1-x^2}\ dx + \int_{1/\sqrt{2}}^1 \sqrt{1-x^2}\ dx.$$

These new integrals are both positive. To see this, make a sketch to see what area they correspond to (pieces of the area of the quarter circle in the first quadrant).

Denote by $A_1 = \int_0^{1/\sqrt{2}} \sqrt{1-x^2}\ dx$ and $A_2 = \int_{1/\sqrt{2}}^1 \sqrt{1-x^2}\ dx$. Then $A = A_1 + A_2$. Recall that we want to show that $\frac{1}{2} \le A$, which is the same as showing that $\frac{1}{2} \le A_1 + A_2$. Note that since both A_1 and A_2 are positive numbers then if we show that $\frac{1}{2} \le A_1$ this will imply that $\frac{1}{2} \le A_1 + A_2$.

We will use the same procedure as before. On the interval $[0, 1/\sqrt{2}]$ we have that $m = 1/\sqrt{2} \le \sqrt{1-x^2} \le 1$, hence

$$\frac{1}{2} \le \int_0^{1/\sqrt{2}} \sqrt{1-x^2}\ dx.$$