TORIC DEGENERATIONS OF FLAG VARIETIES
VIA TROPICAL GEOMETRY

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TORIC DEGENERATIONS

Let X be a projective variety in $\mathbb{P}^n$.

Definition: A toric degeneration of X is a flat family $\phi : \mathcal{F} \mapsto \mathbb{A}^1$, such that $\phi(t)$ is isomorphic to X if $t \neq 0$ and $\phi(0)$ is a toric variety X'.

FLAG VARIETIES

The set of complete flags

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = k^n$$

is an algebraic variety, which we will denote by $\mathcal{F}l_n$, called the complete flag variety. It is naturally embedded in a product of Grassmannians using the Plücker coordinates. It is also isomorphic to SL_n/B.

- Toric degenerations of flag varieties have been studied extensively from the point of view of representation theory. An important ingredient is the string polytope introduced by Littelman and Berenstein-Zelevinsky, later used by Caldero to produce toric degenerations.
- From tropical point of view the problem has been first studied by Speyer and Sturmfels in the case of the simplest flag variety $Gr(2, n)$.

GRÖBNER TORIC DEGENERATIONS

Let k be a field and fix the trivial valuation on it. Consider a Laurent polynomial $f = \sum a_{\boldsymbol{u}} x^{\boldsymbol{u}}$, with $\boldsymbol{u} \in \mathbb{Z}^n$. For each $\boldsymbol{w} \in \mathbb{R}^n$ we define:

- the initial form $in_{\boldsymbol{w}}(f)$ of f with respect to $\boldsymbol{w}$ to be the sum of all monomials of f for which $\boldsymbol{w} \cdot \boldsymbol{u}$ is minimal.
- the initial ideal $in_{\boldsymbol{w}}(I)$ of an ideal I of the Laurent polynomial ring to be the ideal $\langle in_{\boldsymbol{w}}(f) : f \in I \rangle$.

Definition: A Gröbner degeneration of $X = V(I)$ is a flat family $\phi : \mathcal{F} \mapsto \mathbb{A}^1$, such that $\phi(t)$ is isomorphic to X if $t \neq 0$ and $\phi(0)$ is $V(in_{\boldsymbol{w}}(I))$. If $V(in_{\boldsymbol{w}}(I))$ is a toric variety, that is, if $in_{\boldsymbol{w}}(I)$ is a prime binomial ideal, then this is a Gröbner toric degeneration.

Fact: One can always find a Gröbner degeneration.

Fact: The polytope of the toric variety X' in the special fiber is the Newton-Okounkov body corresponding to the pair (A, ν), where A is the homogeneous coordinate ring of X and ν is a Krull valuation, which one can obtain from a cone in the tropicalization of X.

TROPICALIZATION

One way of finding Gröbner toric degenerations is through tropicalization.

Definition: Let $X = V(I)$, where I is an ideal in the Laurent polynomial ring over k. The tropicalization of X is the set

$$trop(X) = \{\boldsymbol{w} \in \mathbb{R}^n : in_{\boldsymbol{w}}(I) \text{ does not contain monomials}\}.$$

Facts:

- The tropicalization $trop(X)$ is a polyhedral complex. In particular there are finitely many cones.
- If $\boldsymbol{w}$ and $\boldsymbol{w'}$ are in the relative interior of the same cone then $in_{\boldsymbol{w}}(I) = in_{\boldsymbol{w'}}(I)$.
- If C is a maximal cone in $trop(X)$ then each initial form has at least two monomials.

TORIC DEGENERATIONS FROM TROPICALIZATION - PRIME CONES

Again let $X = V(I)$, where I is an ideal in the Laurent polynomial ring over k.

When C is a maximal cone in $trop(X)$ such that the initial ideal $in_{\boldsymbol{w}}(I)$ is binomial and prime, then:

- the resulting Gröbner degeneration is a toric degeneration and
- we call the cone C a prime cone.

TORIC DEGENERATIONS FROM TROPICALIZATION - NON-PRIME CONES

Not all maximal cones give rise to toric initial ideals. In the case of $\mathcal{F}l_4$ six maximal cones give rise to binomial but non-prime initial ideals. In the case of $\mathcal{F}l_5$ we also have non-prime and non-binomial initial ideals.

Tropicalization depends on the embedding of the variety, hence a different embedding might result in different toric degenerations.

We can carefully choose a new embedding of $\mathcal{F}l_4$ into a higher dimensional projective space and tropicalize again. There the non-prime cones split into prime ones resulting into new Gröbner toric degenerations. This procedure is not restricted to flag varieties.

RESULTS

We give a summary of the results. We compare the cones and the toric degenerations, more precisely the polytopes of the resulting toric varieties, which we obtain via tropicalization (coming from both the prime and non-prime cones after re-embedding) with the ones obtained by representation theory methods.

- In the case of $\mathcal{F}l_4$ and $\mathcal{F}l_5$ there are Gröbner toric degenerations that are not isomorphic to any of the degenerations associated to string polytopes.
- Some string polytopes are combinatorially equivalent to the polytopes associated to Gröbner toric degenerations.

THEOREM

[Bossinger,Lamboglia,M.,Mohammadi]

- *There are 4 non-isomorphic Gröbner toric degeneration of the flag variety $\mathcal{F}l_4$. Among these 4 there is one which is non-isomorphic to any of the degenerations coming from string polytopes. The polytopes associated to the remaining three degenerations are combinatorially equivalent to string polytopes.*
- *Each of the non-prime cones of $trop(\mathcal{F}l_4)$ gives rise to three new toric degenerations. Moreover, two of the three new polytopes are combinatorially equivalent to the previously missing string polytope.*
- *There are 180 non-isomorphic Gröbner toric degenerations of the flag variety $\mathcal{F}l_5$. Among these there are 168 that are non-isomorphic to the ones coming from string polytopes and 12 have polytopes combinatorially equivalent to string polytopes.*

REFERENCES

- The code needed for this paper is available as a Macaulay2 package `ToricDegenerations`, which can be found at https://github.com/ToricDegenerations
- L.Bossinger,S.Lamboglia,K.Mincheva,F.Mohammadi. *Computing toric degenerations of flag varieties*. Combinatorial Algebraic Geometry. Fields Institute Communications, vol 80. Springer (2017), arXiv 1702.05505.